Approximate symmetries of Hamiltonians

Joint work with Steve Flammia
arXiv:1608.02600

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AIP 2016

Goals

- Use symmetries to indirectly study ground space
- Show non-commuting symmetries imply degeneracy
- Extend analysis to <u>approximate</u> symmetries

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Symmetries

For a system with a Hamiltonian H, an exact symmetry S is a unitary with

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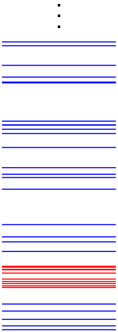
Degeneracy

Start with a Hamiltonian H with a band which is:

- Exactly degenerate
- The ground space
- Gapped

$\sigma(H) =$



Why might we care about degeneracy?

- Serves as crude signal of phases of matter, e.g. topological order.
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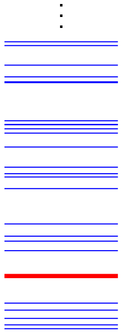
$\sigma(H) =$ 

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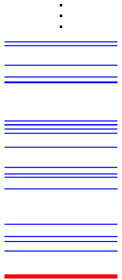
$\sigma(H) =$ 

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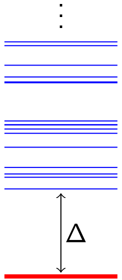

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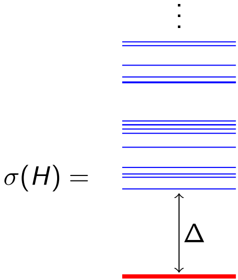

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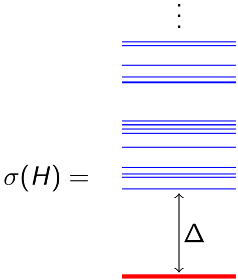

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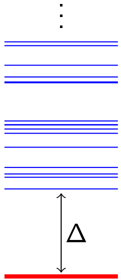
$\sigma(H) =$ 

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Example: Ising Model

$$H = -\sum_i Z_i Z_{i+1}$$

The Quantum Ising model has:

- Two non-commuting symmetries

$$\bar{X} = \prod_i X_i \qquad \bar{Z} = Z_1.$$

- Two-dimensional ground space

$$G = \mathrm{Span}\{|000\ldots\rangle, |111\ldots\rangle\}.$$

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Non-commuting symmetries

To certify degeneracy we want non-commuting symmetries.

For some phase η, the twisted commutator is defined

$$[U, V]_\eta := UV - \eta VU.$$

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Restriction to groundspace

Lemma

For any approximate symmetry

$$\|[U, H]\| \leq \epsilon,$$

there exists a unitary u on the ground space, which approximates the action of U

$$\|u - \Pi U \Pi\| \leq 3\epsilon/\Delta,$$

where Π is the ground space projector.

This allows us to restrict to the ground space with low distortion

$$\|[U, H]\|, \|[V, H]\| \leq \epsilon, \; \|[U, V]_\eta\| \leq \delta \qquad \Longrightarrow \qquad \|[u, v]_\eta\| \leq \delta' := \delta + 12\epsilon/\Delta$$

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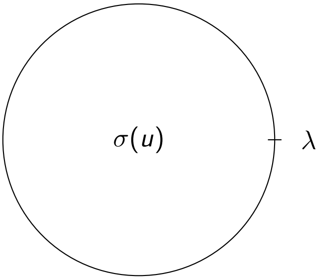
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u, v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u \left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



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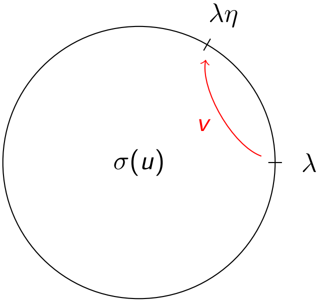

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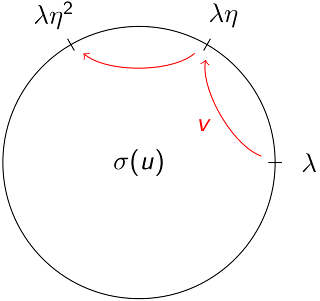

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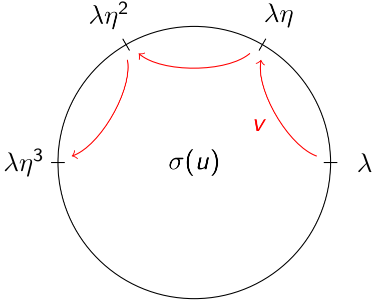

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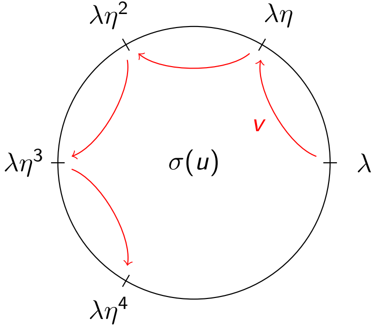

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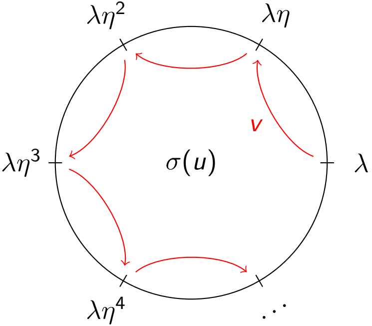

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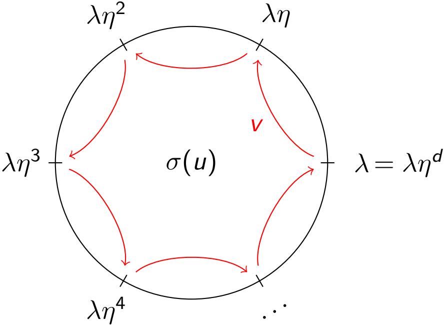

Approximate case

If we relax

$$[u,v]_\eta = 0 \qquad \to \qquad \|[u,v]_\eta\| \leq \delta'$$

then once again starting from λ-eigenvector and considering the action of v:

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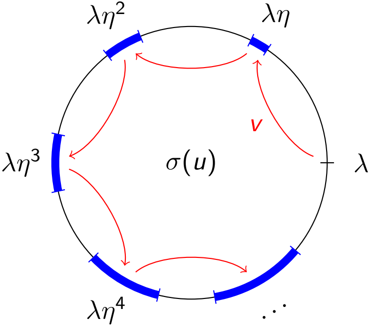
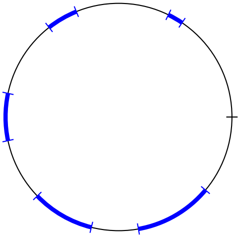


Full and partial certification

For η a dth root of unity

$$\|[u, v]_\eta\| < \frac{2}{d-1}\left[1 - \cos\pi/d\right]$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



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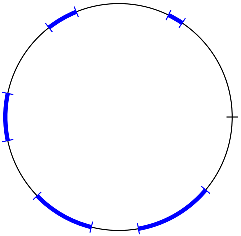

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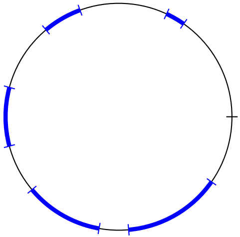

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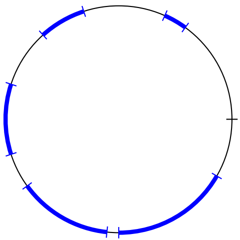

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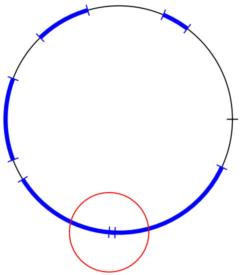

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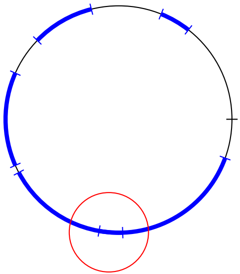

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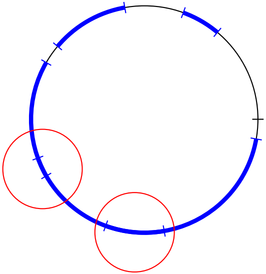

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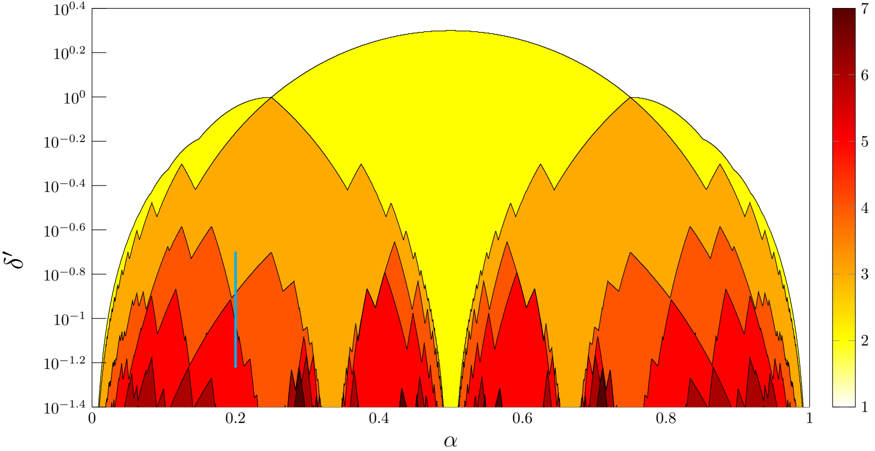
Certifiable degeneracy

If we have $\|[u,v]_\eta\| \le \delta'$ for $\eta := e^{2i\pi\alpha}$, then the degeneracy we can certify is



Conclusion

- Non-commuting symmetries can serve as certificates of degeneracy
- Twisted commutation gives provable degeneracy certification
- These certifcates are valid even with approximate commutation relations
- In the full paper (arXiv:1608:02600) we were able to extend these results to more general norms and more general bands

I will also be presenting a poster on this work, number 229 on Tuesday night (1800-2000).