Approximate symmetries as witnesses of ground space degeneracy

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Introduction

Symmetries are powerful tool to indirectly probe properties of physical systems. In this work we demonstrate how approximate symmetries can witness ground space degeneracy. Given several such symmetries of a Hamiltonian satisfying certain approximate commutation relations, we bound the allowed dimensions of a gapped ground space. Importantly, the bounds we require to bound the degeneracy are all independent of the dimension of the full Hilbert space.

Reduction to the ground space

To use approximate symmetries to study the properties of the ground space, we start by showing there exist nearby operators which exactly preserve the ground space.

Theorem: Restriction to the ground

Suppose you have a Hamiltonian H, which has a spectral gap of Δ and ground space projector Π. For any unitary $\tilde{U}$ with

$$\left\|\left[H,\tilde{U}\right]\right\| \leq \epsilon$$

then there exists a nearby unitary U such that

$$\left\|U-\tilde{U}\right\| \leq 24\epsilon/\Delta \quad \text{and} \quad [U,\Pi]=0.$$

Using this, we can pull approximate symmetries down into the ground space, such that commutation relations between them are preserved with low distortion. By simply studying the relationship between commutation relations and dimensions, these operators can be used as degeneracy witnesses.

Twisted commutator

The commutation relation we are going to study is the α-*twisted commutator*. For some $\alpha \in \mathbb{R}$ and matrices X and Y, it is defined

$$[X,Y]_\alpha := XY - e^{2\pi i\alpha}YX.$$

Note for $\alpha=0$ and $\alpha=1/2$ this reduces to the standard commutator and anti-commutator respectively.

In the exact case, the link between dimensionality and twisted commutation is given by the finite-dimensional Stone-von Neumann Theorem [1].

Stone-von Neumann Theorem

For any coprime integers p and q, and unitaries U and V,

$$[U,V]_{p/q}=0$$

implies $\dim U = \dim V \propto q$.

Our results can be consider approximate generalisations of Stone-von Neumann.

Lower bound: Single pair

The first result we have is a lower bound on the degeneracy, based on the magnitude of the twisted commutator. Consider a pair of unitaries U and U with

$$\left\|[U,V]_{1/d}\right\| \leq \delta$$

for some $d \in \mathbb{N}$. If we let $\eta := e^{2\pi i\alpha}$, then this gives that for any state $|\phi\rangle$

$$\left|\left\langle\phi\middle|V^\dagger UV\middle|\phi\right\rangle - \eta\left\langle\phi|U|\phi\right\rangle\right| \leq \delta.$$

We can see therefore that V maps eigenstates of U to approximate eigenstates, of value rotated by η. Iterating this we can contruct an approximate eigenbasis, a sketch of which is shown in Figure 1.

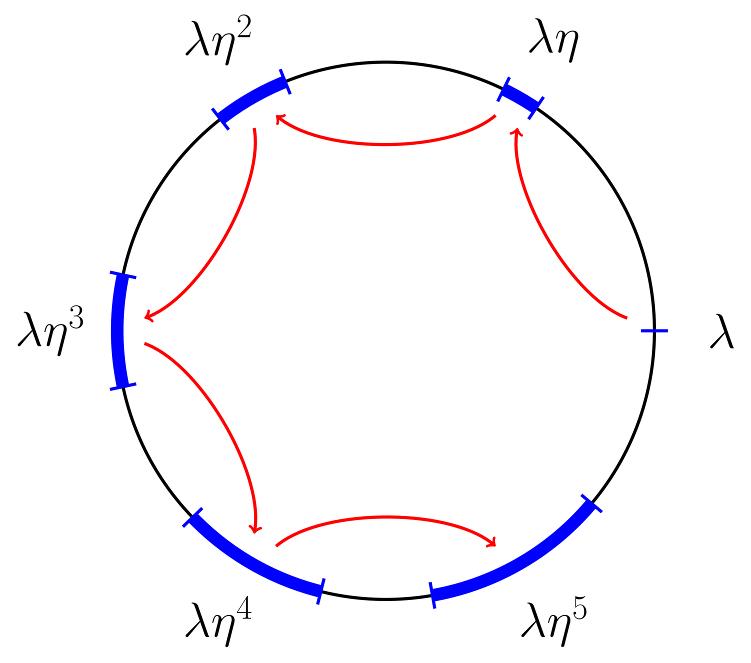

Figure 1: The action of V on the eigenspaces of U, for the example of $\alpha=1/6$. Assuming an eigenvalue at λ, the above argument implies the existence of an eigenvalue lying in each blue arc.

If all of the arcs are disjoint, then clearly we have at least q eigenvalues, and therefore can lower bound the dimension by q.

Theorem: Lower bound 1

For a fixed $d \in \mathbb{N}$, any two unitaries satisfying

$$\left\|[U,V]_{1/d}\right\| < \frac{1}{d-1}\left[1-\cos\pi/d\right]$$

are at least d-dimensional.

More generally if we allow for the arcs to overlap, the certifiable degeneracy is given in Figure 2

Lower bound: Double pair

By extending our analysis to *approximate shared eigenvectors* [2], we can also consider the case of two approximately commuting pairs of approximate symmetries.

Theorem: Lower bound 2

For fixed $d_1, d_2 \in \mathbb{N}$ with $d_1 \leq d_2$, any unitaries satisfying the commutation relations

$$\left\|[U_1,V_1]_{1/d_1}\right\| \leq \delta, \qquad \left\|[U_2,V_2]_{1/d_2}\right\| \leq \delta,$$

$$\|[U_1,U_2]\| \leq \delta^2, \quad \|[U_1,V_2]\| \leq \delta, \quad \|[U_2,V_1]\| \leq \delta,$$

for

$$\delta < \frac{\sin^2(\pi/2d_1)}{(d_1+d_1d_2+d_2)(d_1d_2-1)^2} \sim \frac{1}{d_1^5 d_2^3}$$

are at least d_1d_2-dimensional.

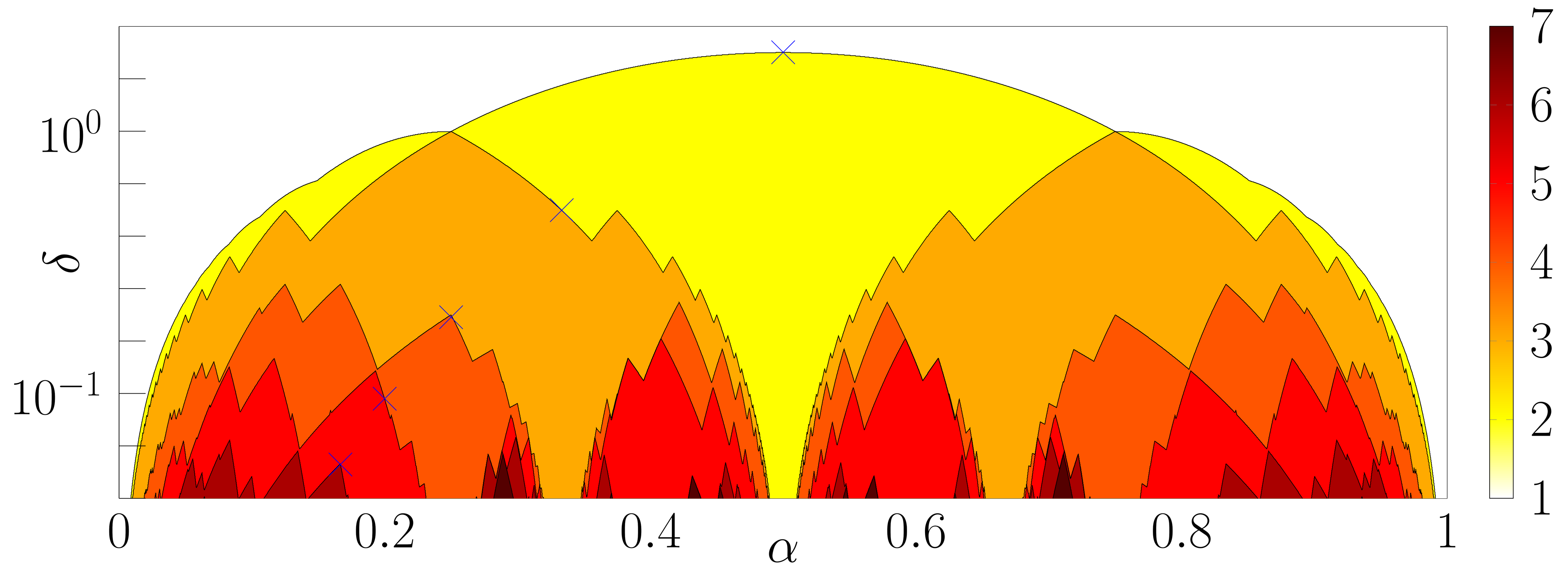

Figure 2: The dimensional lower bounds which can be certified by a twisted commutator $\|[U,V]_\alpha\| \leq \delta$. The results of Lower bound 1 are indicated by the blue crosses.

Minimum twisted commutator

Restricting to the case of a single pair of symmetries, we can strengthen our analysis to a complete analysis of the disallowed dimensions. To do this we find an explicit minimum for the twisted commutator, as a function of both twisting and dimension, i.e.

$$\Lambda_{\alpha,g} := \min_{u,v\in U(g)} \|[u,v]_\alpha\|_p.$$

Any operators with a twisted commutator smaller than $\Lambda_{\alpha,g}$ therefore cannot be g-dimensional.

The main technical tool needed is the Wieland-Hoffman inequality [3], which lower bounds the distance between two matrices by the distance between their spectra.

Wielandt-Hoffman Inequality

If we let $\lambda_i(X)$ denote the ith eigenvalue of X (arbitrarily ordered), then

$$\|A-B\|_F^2 \geq \min_\sigma \left|\lambda_i(A)-\lambda_{\sigma(i)}(B)\right|^2,$$

where σ is a permutation over the indices.

Using the Wielandt-Hoffman, we can lower bound $\Lambda_{\alpha,g}$. Considering the generalised Pauli operators it can be seen that this bound is tight.

Theorem: Minimum twisted comm.

For any unitaries g-dimensional u and v,

$$\|[u,v]_\alpha\| \geq 2\sin\left(\pi\left|\frac{g\alpha-\lfloor g\alpha\rceil}{g}\right|\right).$$

A plot of the above minimum is given in Figure 3.

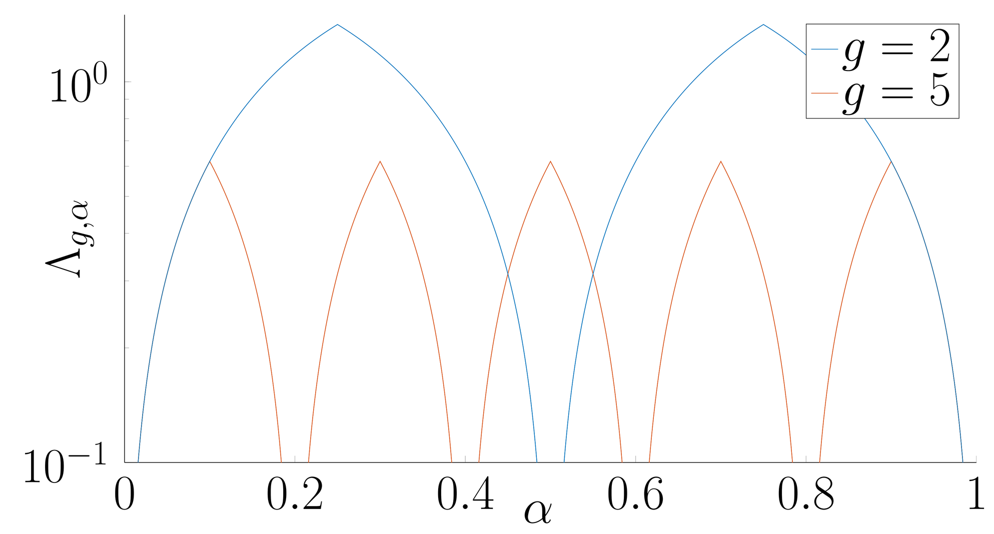

Figure 3: Minimum twisted commutator magnitude. Note that it vanishes as multiples of $1/g$.

Application to topological systems

A recent paper by Bridgeman et. al. in Ref. [4] aims to numerically optimise cost functions involving twisted commutators, with the aim of classifying the topological order of condensed matter systems. In the case of several exactly solvable models they consider, the theory outlined above predicts their findings to within numerical error.

References

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[4] J. C. Bridgeman, S. T. Flammia, and D. Poulin, "Detecting Topological Order with Ribbon Operators," arXiv:1603.02275, (2016).