Approximate symmetries of Hamiltonians

Joint work with Steve Flammia (USyd/MIT)
arXiv:1608.02600

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TQC 2017

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Ground space degeneracy

Why?

- Crude signal of topological order[1]
- Code size of a quantum code[2]

How?

- Find orthogonal ground states (ED, DMRG, PEPS, QMC, etc.)
- Indirect certificates such as symmetries

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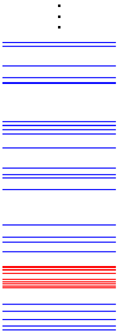
Degeneracy

H is a Hamiltonian, with a band which is:

- Exactly degenerate
- The ground space
- Gapped

In the full paper we consider general gapped bands, and infinite dimensional Hamiltonians.

$\sigma(H) =$



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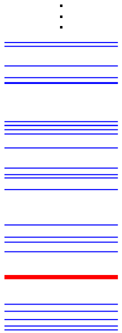

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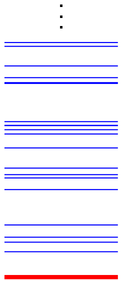
$\sigma(H) =$ 

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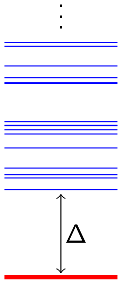

Twisted commutation

To certify degeneracy we want non-commuting symmetries. Specifically we will consider symmetries which twisted commute

$$[U, V]_\alpha := UV - e^{2\pi i\alpha} VU = 0.$$

for $\alpha \in [0, 1)$.

We are also going to consider relaxing this to

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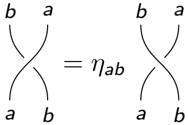
Twisted commutation and Abelian anyons

Braiding two Abelian anyons incurs an overall phase factor.



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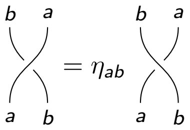

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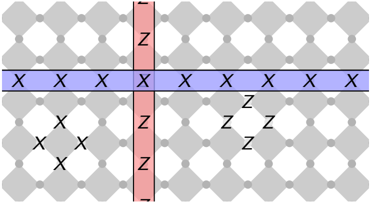

Restriction to ground space

Lemma

For any approximate symmetry $\|[U,H]\| \leq \epsilon$, there exists a unitary u on the ground space, which approximates the action of U

$$\|u - \Pi U \Pi\| \leq (\epsilon/\Delta)^2,$$

where Π is the projector onto the ground space.

This allows us to restrict to the ground space with low distortion

$$\|[U,H]\|, \|[V,H]\| \leq \epsilon,\ \|[U,V]_\alpha\| \leq \delta \quad \implies \quad \|[u,v]_\alpha\| \leq \delta' := \delta + 6(\epsilon/\Delta)^2.$$

A similar result holds for any unitarily invariant norm.

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Degeneracy certification

Now that we can restrict to the ground space, we can bound the degeneracy by now exploring the relationship between dimensionality and twisted commutation.

We will do this in two stages:

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Exact case

Stone-Von Neumann Theorem (finite dimension)

If $[u,v]_{p/q} = 0$ for p and q coprime, then $\dim(u)$ is a multiple of q.

Let $\eta := e^{2\pi i p/q}$ and consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u\left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$

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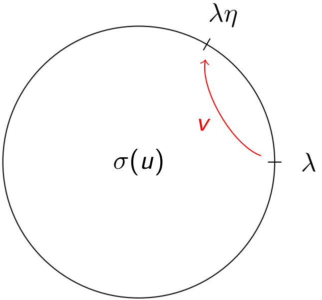

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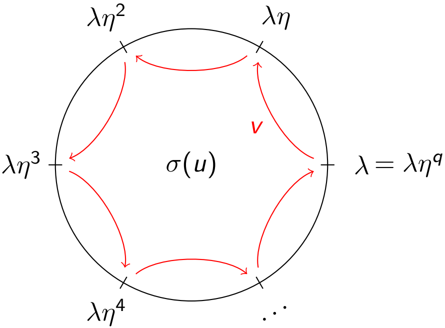


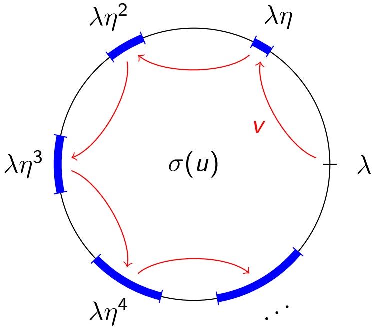
Approximate case

If we relax

$$[u, v]_\alpha = 0 \qquad \rightarrow \qquad \|[u, v]_\alpha\| \leq \delta'$$

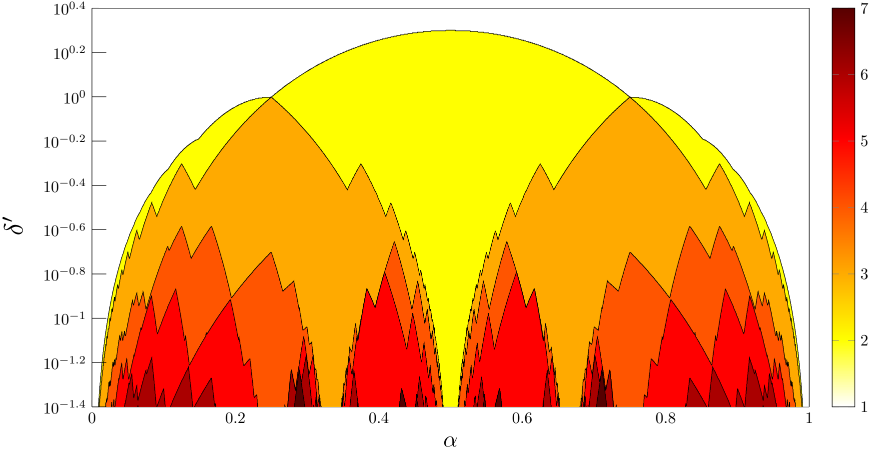
then once again starting from λ-eigenvector and considering the action of v:

$$|\langle\lambda|v^\dagger uv|\lambda\rangle - \lambda\eta| \leq \delta'$$



Minimum degeneracy

If we have $\|[u,v]_\alpha\| \le \delta'$, then the degeneracy is at least



Two pairs

We can also consider two twisted pairs $(u_1, v_1), (u_2, v_2)$ with twisted commutation relations

$$\left\| [u_1, v_1]_{1/d} \right\| \leq \delta \qquad \text{and} \qquad \left\| [u_2, v_2]_{1/d} \right\| \leq \delta,$$

for some integer d, and commutation relations

$$\left\| [u_1, u_2] \right\| \leq \delta^2, \qquad \left\| [u_1, v_2] \right\| \leq \delta, \qquad \left\| [u_2, v_1] \right\| \leq \delta.$$

Two pairs

Here we will find approximate shared eigenvectors of (u_1, u_2)

$$\left|\langle\lambda_1,\lambda_2|u_1|\lambda_1,\lambda_2\rangle-\lambda_1\right|\leq\zeta_1 \qquad \text{and} \qquad \left|\langle\lambda_1,\lambda_2|u_2|\lambda_1,\lambda_2\rangle-\lambda_2\right|\leq\zeta_2.$$

We now find that the vectors (λ_1, λ_2) lie in rectangles on an torus.

$$\delta \leq \frac{\sin^2(\pi/2d)}{d(d+2)(d^2-1)^2} \sim \frac{1}{d^6} \qquad \Longrightarrow \qquad \dim \geq d^2$$

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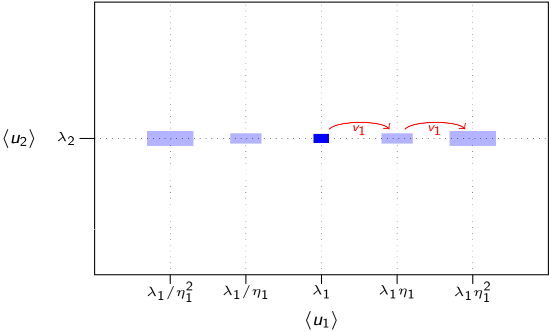

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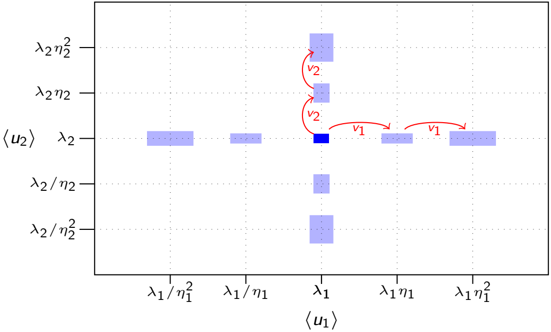

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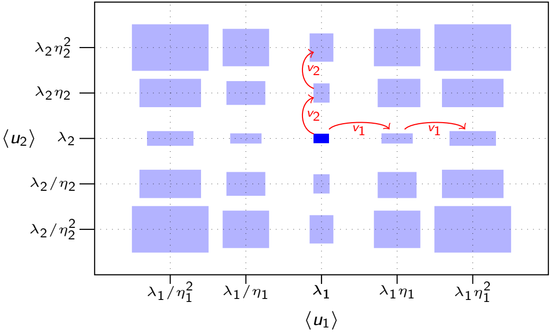

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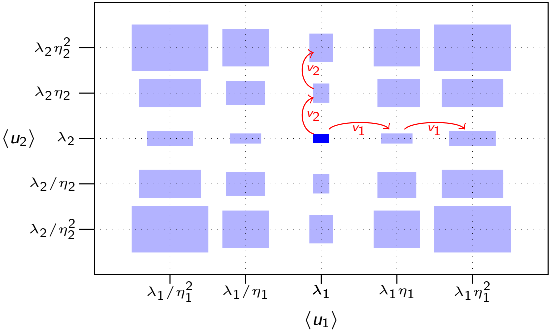

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Minimum twisted commutator

We know that $[u, v]_{p/q} = 0$ iff $\dim(u) \propto q$, but what if $\| [u, v]_{p/q} \| \ll 1$?

What are the allowed values of $\| [u, v]_{p/q} \|$ are in a given dimension g.

Specifically, we will look at minimum twisted commutator in a given dimension g

$$\Lambda_{\alpha,g} := \min_{u,v \in U(g)} \| [u, v]_\alpha \|$$

We also bound similar minima for all Schatten-Ky Fan norms $\| \cdot \|_{(p,k)}$ for $p \geq 2$.

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Wielandt-Hoffman

The main technical tool comes from spectral perturbation theory.

Theorem (Wielandt-Hoffman theorem)

For normal matrices a and b, the Frobenius distance is lower bounded by the spectral Frobenius distance

$$\|a-b\|_2 \geq \min_{\sigma \in S_g} \left\| \sigma[\vec{\lambda}(a)] - \vec{\lambda}(b) \right\|_2 .$$

$\vec{\lambda}(x)$ denotes the eigenvalues of x, and S_g the symmetric group on g objects.

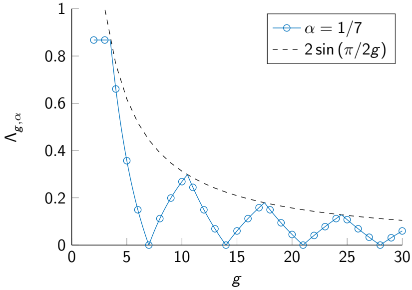
Minimum twisted commutator

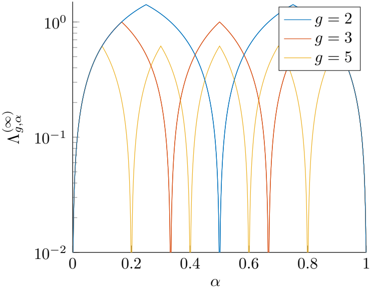
Minimising this quantity we find that

$$\|[u,v]_\alpha\| \geq 2\sin\left(\pi\left|\frac{\lfloor g\alpha\rceil - g\alpha}{g}\right|\right),$$

which is saturated by appropriate powers of the g-dimensional Paulis

$$u = X_g, \qquad v = Z_g^{\lfloor g\alpha\rceil}.$$





Conclusion and further work

- Non-commuting symmetries can serve as certificates of degeneracy
- Twisted commutation gives provable degeneracy certification
- These certificates are robust to approximate commutation relations

- Can we bridge the gap between numerics and analytics for non-exactly solvable models? What about non-Abelian anyons?
- What happens if we take in to account locality?
- Do these arguments generalise to symmetric protected topological order?
- Can this be extended to more general approximate representations?
- Do provably efficient algorithms for calculating these certificates exist?

ArXiv: 1608.02600