Beyond the thermodynamic limit: finite-size corrections to state interconversion rates

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Introduction

An important problem in thermodynamics is which operations can be performed, given access to a thermal bath of a specific temperature. In this work we consider the problem of state interconversion, giving finite-size corrections in the presence of non-zero errors.

Thermal operations

The set of allowed operations we will consider are thermal operations. Consider a system with Hamiltonian H, and a bath with Hamiltonian H_B and thermal state γ_β, at some inverse-temperature β. A *thermal operation* is a channel $\mathcal{E}^\beta$ of the form

$$\mathcal{E}^\beta(\rho) = \mathrm{Tr}_B \left(U \left(\rho \otimes \gamma_\beta \right) U^\dagger \right), \tag{1}$$

where U is an unitary acting on the system and the bath, which is energy preserving, $[U, H + H_B] = 0$.

Energy incoherence

In this work we will not consider arbitrary states, but only energy incoherent states ρ, such that $[\rho, H] = 0$. These state generally lie strictly between classical states and general quantum states, reducing to these in the case where the Hamiltonian is non-degenerate and scalar respectively.

State interconversion

Consider a system with Hamiltonian H, an inverse temperature β, and initial and target states ρ and σ. We define the optimal interconversion rate $R^*(n, \epsilon)$ as the maximum rate R at which there exists a thermal operation $\mathcal{E}^\beta$ which sends $\rho^{\otimes n}$ to $\sigma^{\otimes Rn}$, to within a fidelity error at most ϵ.

References

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Majorisation

For energy incoherent states, whether or not an input state can be converted into a target state is equivalent to a majorisation condition. If we consider $\beta = 0$, then ρ can be converted into σ if and only if $\mathrm{eig}(\rho) \succ \mathrm{eig}(\sigma)$, where $\mathrm{eig}(\cdot)$ gives a vector of the eigenvalues, and $p \succ q$ denotes majorisation, defined as

$$\sum_{i=1}^k p_i^\downarrow \geq \sum_{i=1}^k q_i^\downarrow \quad \forall k, \tag{2}$$

where $x_i^\downarrow$ denotes the ith largest entry of a vector x.

For $\beta \neq 0$ there exists an analogous concept known as *thermo-majorisation* [3].

Approximate majorisation

To study state interconversion with non-zero errors, we need an approximation notion of majorisation. We define two such notions.

A distribution p *ϵ-pre-majorises* q, which we denote $p\,{}_\epsilon\!\succ q$, if and only if there exists a $\tilde{p}$ such that $\tilde{p} \succ q$ and $\delta(p, \tilde{p}) \leq \epsilon$, where $\delta(\cdot,\cdot)$ denotes infidelity. Similarly p *ϵ-post-majorises* q, denoted $p \succ_\epsilon q$, if there exists a $\tilde{q}$ such that $p \succ \tilde{q}$ and $\delta(q, \tilde{q}) \leq \epsilon$.

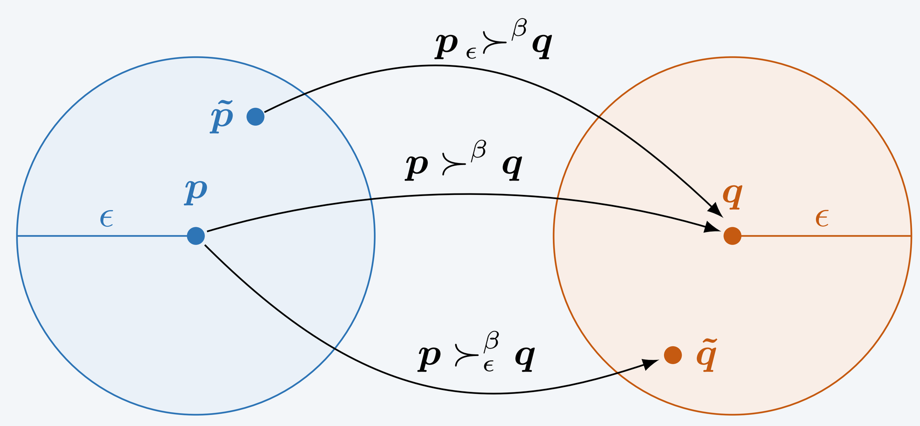


Figure 1: Pre and post majorisation. Both circles denote infidelity balls, and arrows indicate majorisation.

Lemma: Pre=post

Pre- and post-majorisation are in fact equivalent, with the same error parameter, i.e.

$$p\,{}_\epsilon\!\succ q \quad \Leftrightarrow \quad p \succ_\epsilon q. \tag{3}$$

Rayleigh-normal distribution

In our bounds we use a family of distributions which interpolate between the Rayleigh and normal distributions, as defined in Ref. [2].

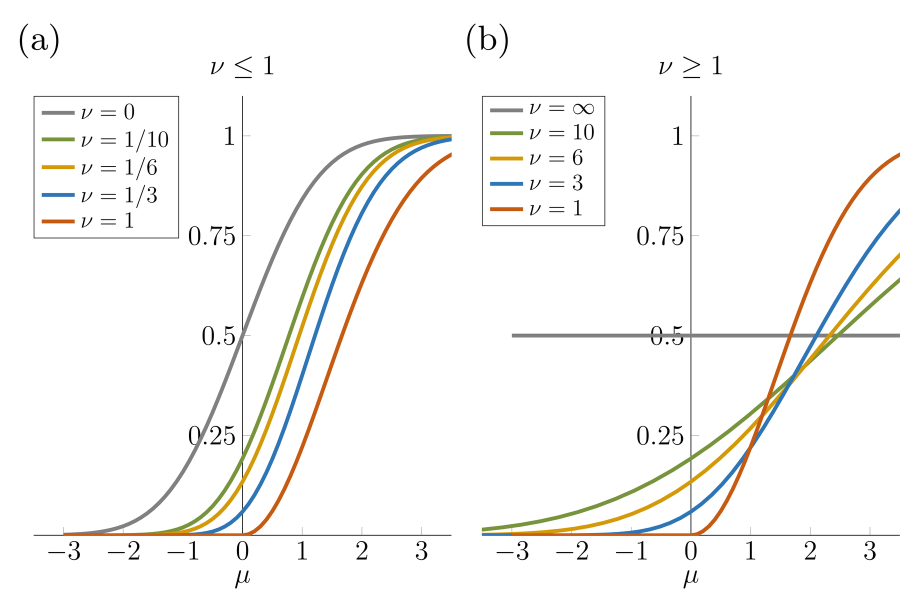


Figure 2: The cumulative distribution function $Z_\nu(\mu)$ for the Rayleigh-normal distributions, for parameter values (a) $\nu \leq 1$ and (b) $\nu \geq 1$.

Theorem: Second-order expansion

The optimal interconversion rate has the second-order expansion

$$R^*(n, \epsilon) \simeq \frac{D(\rho\|\gamma)}{D(\sigma\|\gamma)} + \sqrt{\frac{V(\rho\|\gamma)}{n D(\sigma\|\gamma)^2}} Z^{-1}_{1/\nu}(\epsilon), \tag{4}$$

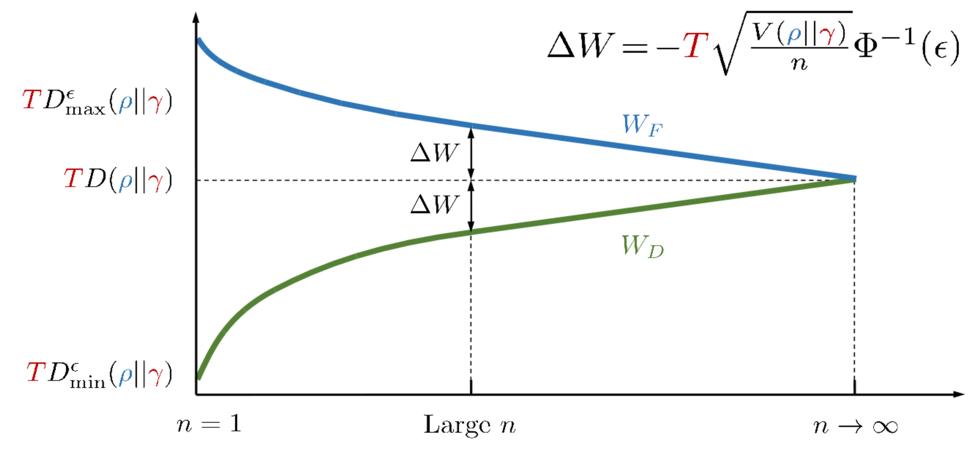
where $\simeq$ denotes equality up to $o(1/\sqrt{n})$, and the *relative entropy* $D(\cdot\|\cdot)$ and *relative entropy variance* $V(\cdot\|\cdot)$ are defined

$$D(\rho\|\sigma) := \mathrm{Tr}\,\rho \left(\log \rho - \log \sigma \right), \tag{5}$$

$$V(\rho\|\sigma) := \mathrm{Tr}\,\rho \Big(\log \rho - \log \sigma - D(\rho\|\sigma) \Big)^2. \tag{6}$$

Work of formation/distillation

One application of our results is to consider converting between a state ρ and an energy eigenstate. This gives us the work of formation/distillation, which we denote W_F and W_D respectively, giving a bound on the discrepancy between these two quantities for large but finite system sizes. Unlike in the single-shot regime, where these quantities are given by the min- and max-entropies [4], here our works of formation and distillation are symmetric around their asymptotic value.



Numerical results

Using an algorithm presented in Ref. [5], the optimal interconversion rates can be numerically calculated for small system sizes.

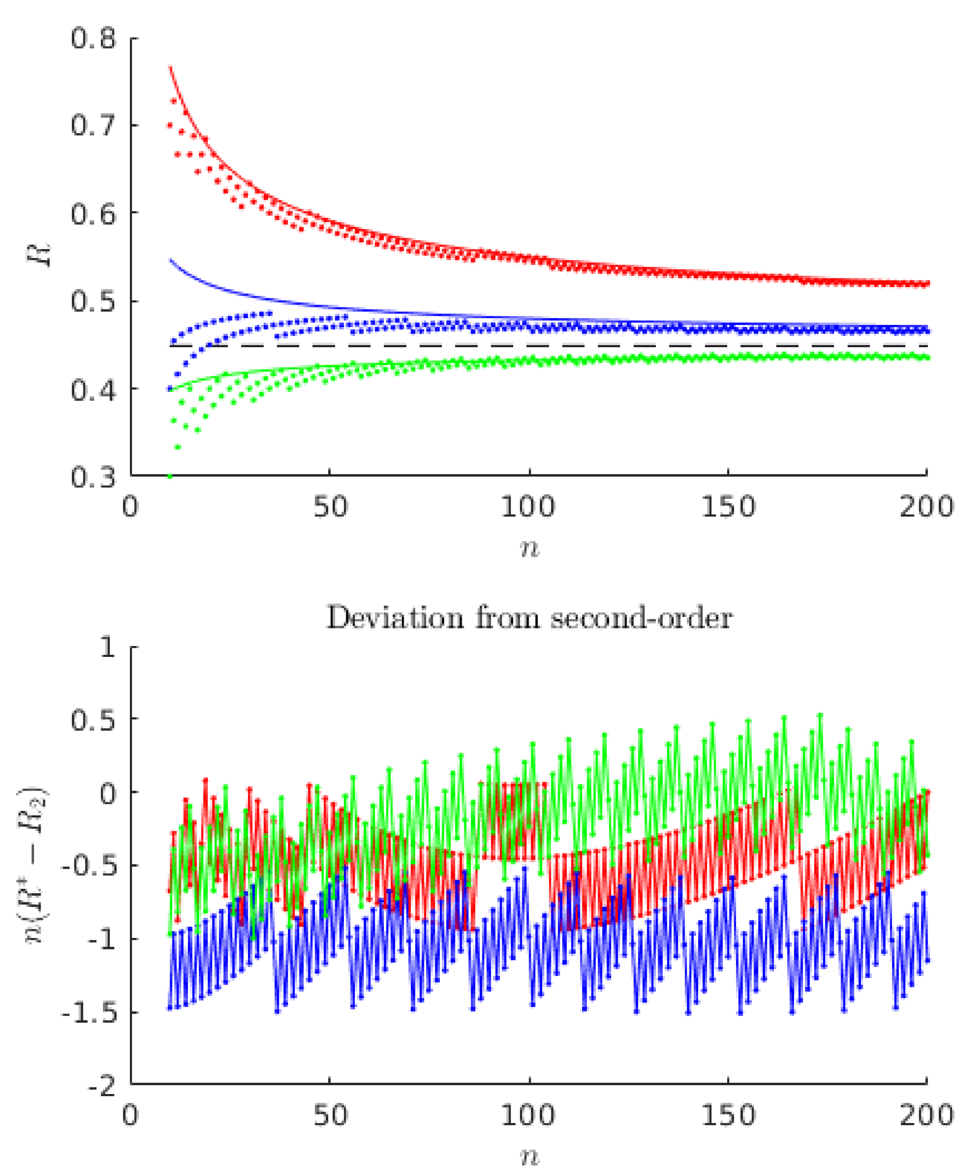


Figure 3: Comparison of the exact interconversion rates with the second order expansion, for two randomly chosen qutrit states. Colour indicates error levels, with red, blue and green corresponding to $\epsilon = 10^{-2}, 10^{-3}, 10^{-8}$ respectively. (Top) Dots indicate exact interconversion rates, and the lines our second order expansions. (Bottom) The deviation from the second-order expansion. These lines being bounded from above and below suggest the third-order term can be strengthened from $o(1/\sqrt{n})$ to $\mathcal{O}(1/n)$.