Statistical mechanical models for quantum codes subject to correlated noise

Christopher Chubb (USyd), Steve Flammia (USyd/Yale)

(me@)christopherchubb.com

arXiv:1809.10704

AIP
2018-12

Quantum codes

Two important questions about quantum codes:

- How do I decode?
- What is the threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

[1] Dennis, Kitaev, Landahl, Preskill, JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`

Quantum codes

Two important questions about quantum codes:

- How do I decode optimally?
- What is the threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

[1] Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

Quantum codes

Two important questions about quantum codes:

- How do I decode optimally?
- What is the fault-tolerant threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

[1] Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

Quantum codes

Two important questions about quantum codes:

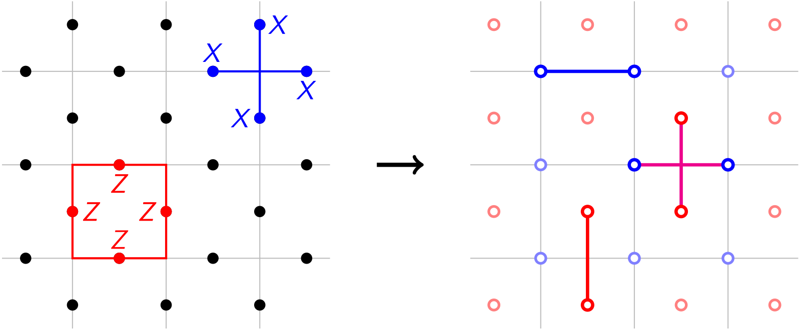
- How do I decode optimally?
- What is the fault-tolerant threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

[1] Dennis, Kitaev, Landahl, Preskill, JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`

Statistical mechanical mapping

The idea here it to construct a family of statistical mechanical models, whose thermodynamic properties reflect the error correction properties of the code.



Toric code → Eight-vertex model

This will allow us to use the analytic and numerical tools developed to study stat mech systems to study quantum codes.

Statistical mechanical mapping

Stabiliser code & Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Error-correction statistics	$\longleftrightarrow$	Thermodynamic statistics

Statistical mechanical mapping

Stabiliser code & Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Threshold	$\longleftrightarrow$	Phase transition
Decoding	$\longleftrightarrow$	Minimising free energy

Our results

- General stat mech mapping construction for arbitrary codes and correlated noise
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Extend to spatio-temporal correlations, such as circuit noise
- Give efficient TN (approximate) maximum likelhood decoders, generalising BSV

Our results

- General stat mech mapping construction for arbitrary codes and correlated noise
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Extend to spatio-temporal correlations, such as circuit noise
- Give efficient TN (approximate) maximum likelhood decoders, generalising BSV

Our results

- General stat mech mapping construction for arbitrary codes and correlated noise
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Extend to spatio-temporal correlations, such as circuit noise
- Give efficient TN (approximate) maximum likelhood decoders, generalising BSV

Our results

- General stat mech mapping construction for arbitrary codes and correlated noise
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Extend to spatio-temporal correlations, such as circuit noise
- Give efficient TN (approximate) maximum likelihood decoders, generalising BSV

Structure of the (independent) mapping

The stat mech mapping gives a bond-disordered Ising/Potts-type model with:

States	$\longrightarrow$	Stabilisers
Disorder	$\longrightarrow$	Error
Couplings	$\longrightarrow$	Single-site Paulis
Coupling strengths	$\longrightarrow$	Fourier transform of error model

For a stabiliser code generated by $\{S_k\}_k$,

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k:[\![\sigma, S_k]\!]=-1} s_k}^{\text{DoF}},$$

where $s_k = \pm 1$, and $[\![A, B']\!]$ is the scalar commutator of Paulis, $AB =: [\![A, B]\!]\, BA$.

Structure of the (independent) mapping

The stat mech mapping gives a bond-disordered Ising/Potts-type model with:

States	$\longrightarrow$	Stabilisers
Disorder	$\longrightarrow$	Error
Couplings	$\longrightarrow$	Single-site Paulis
Coupling strengths	$\longrightarrow$	Fourier transform of error model

For a stabiliser code generated by $\{S_k\}_k$,

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k:[\![\sigma, S_k]\!]=-1} s_k}^{\text{DoF}},$$

where $s_k = \pm 1$, and $[\![A, B']\!]$ is the scalar commutator of Paulis, $AB =: [\![A, B]\!] \, BA$.

Structure of the (independent) mapping

The stat mech mapping gives a bond-disordered Ising/Potts-type model with:

States	$\longrightarrow$	Stabilisers
Disorder	$\longrightarrow$	Error
Couplings	$\longrightarrow$	Single-site Paulis
Coupling strengths	$\longrightarrow$	Fourier transform of error model

For a stabiliser code generated by $\{S_k\}_k$,

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma\in\mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k:[\![\sigma,S_k]\!]=-1} s_k}^{\text{DoF}},$$

where $s_k = \pm 1$, and $[\![A, B']\!]$ is the scalar commutator of Paulis, $AB =: [\![A, B]\!]\, BA$.

Independent case: Properties

The stabiliser group is encoded as a gauge symmetry of the form

$$s_k \to -s_k \qquad \text{and} \qquad E \to ES_k$$

which implies that $Z_E = Z_{ES_k}$.

If the coupling strengths satisfy the Nishimori condition, then

$$e^{-\beta H_E(\vec{0})} = \Pr(E) \qquad \Longrightarrow \qquad Z_E = \Pr(\overline{E}) := \sum_S \Pr(ES).$$

Nishimori condition: $\beta J_i(\sigma) = \frac{1}{4}\sum_{\tau\in\mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket$,

This intrinsically links the thermodynamic and error correction statistics.

Independent case: Properties

The stabiliser group is encoded as a gauge symmetry of the form

$$s_k \to -s_k \qquad \text{and} \qquad E \to ES_k$$

which implies that $Z_E = Z_{ES_k}$.

If the coupling strengths satisfy the Nishimori condition, then

$$e^{-\beta H_E(\vec{0})} = \Pr(E) \qquad \Longrightarrow \qquad Z_E = \Pr(\overline{E}) := \sum_S \Pr(ES).$$

Nishimori condition: $\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket ,$

This intrinsically links the thermodynamic and error correction statistics.

Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

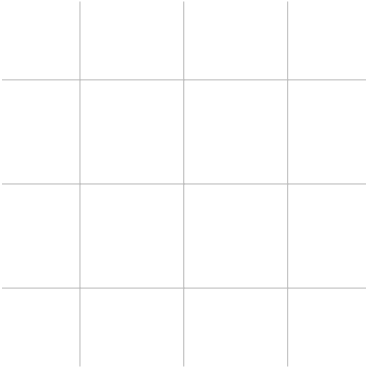
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

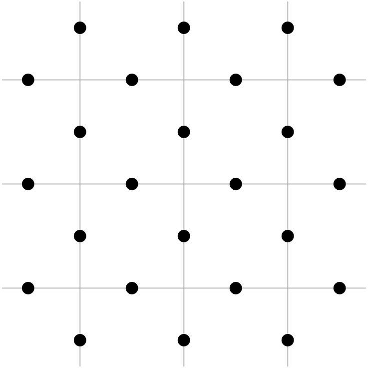
Step 2: Interactions

$$H_I = -\sum_{v\sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v\sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

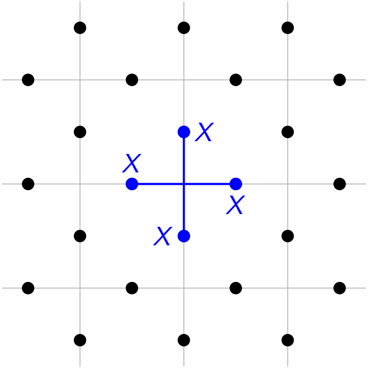
Step 2: Interactions

$$H_I = -\sum_{v\sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v\sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips $\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

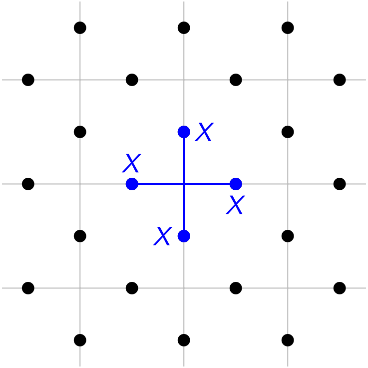
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

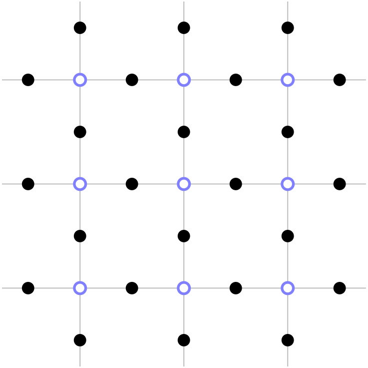
Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \qquad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v



Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$

Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

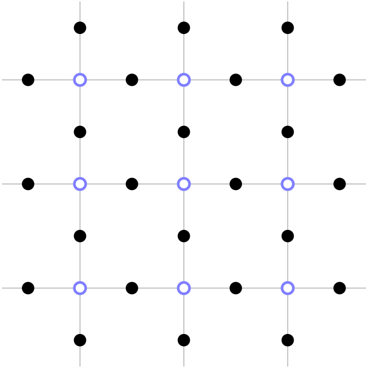
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

where $e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \qquad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

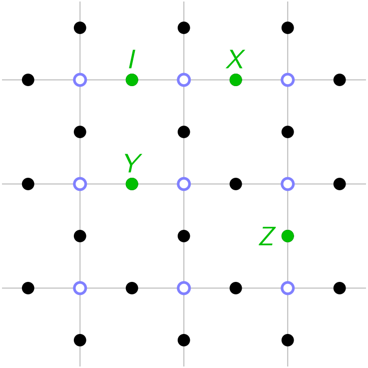
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

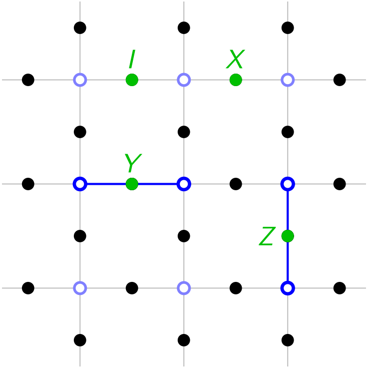
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips $\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

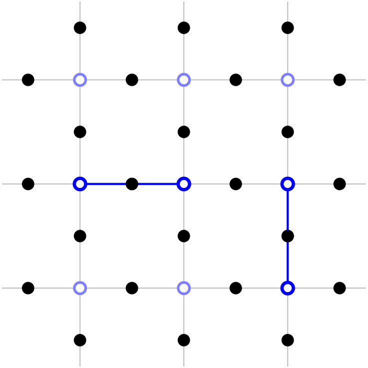
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

where $e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \qquad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

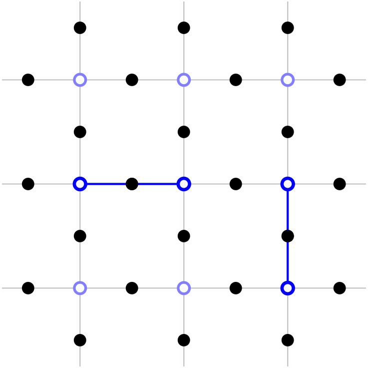
Step 2: Interactions

$$H_I = -\sum_{v\sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v\sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$

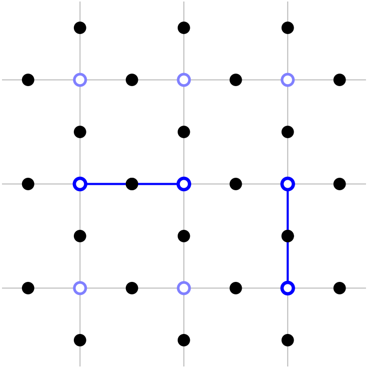


Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$



Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

where $e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$

Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

Step 2: Interactions

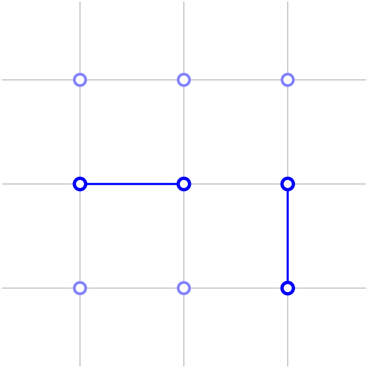
$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

where $e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$

$\Pr(+J) = p, \quad \Pr(-J) = 1 - p.$

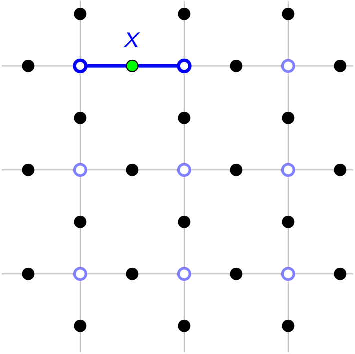


$\pm J$ Random-bond Ising Model

Other independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



Colour code

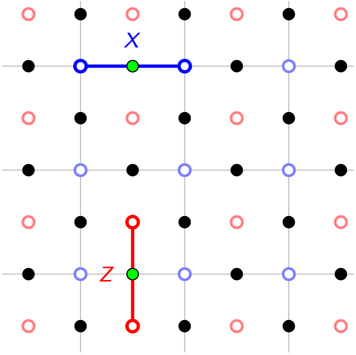
Bit-flip → Random 3-spin Ising
Indep. $X\&Z$ → 2×Random 3-spin Ising
Depolarising → Random interacting 8-vertex[2]

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2] Bombin et.al., PRX 2012, `doi:10/crz5`, `arXiv:1202.1852`

Other independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



Colour code

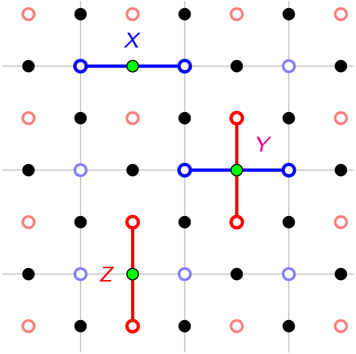
Bit-flip → Random 3-spin Ising
Indep. $X\&Z$→ 2×Random 3-spin Ising
Depolarising → Random interacting 8-vertex[2]

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2] Bombin et.al., PRX 2012, `doi:10/crz5`, `arXiv:1202.1852`

Other independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



Colour code

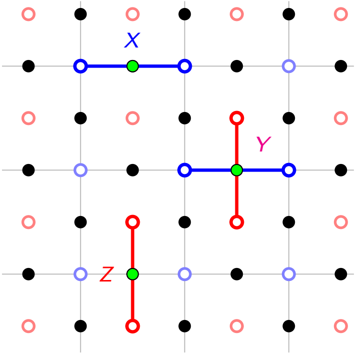
Bit-flip → Random 3-spin Ising
Indep. $X\&Z$→ 2×Random 3-spin Ising
Depolarising → Random interacting 8-vertex[2]

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2] Bombin et.al., PRX 2012, `doi:10/crz5`, `arXiv:1202.1852`

Other independent examples

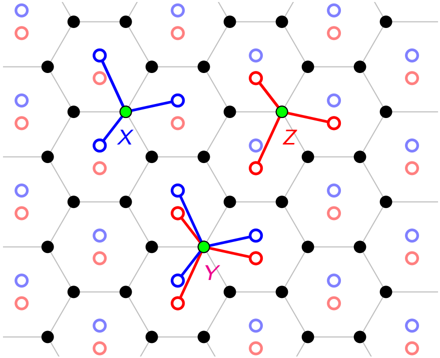
Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



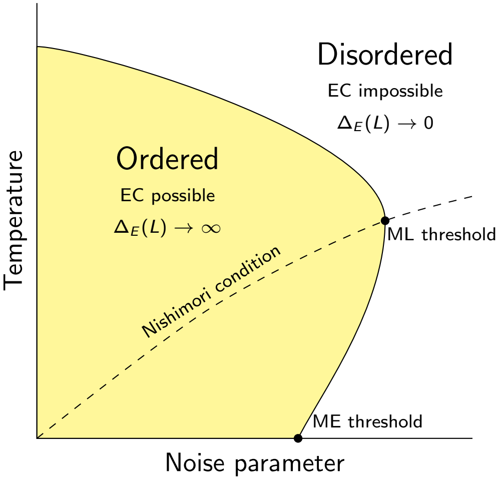
Colour code

Bit-flip → Random 3-spin Ising
Indep. $X\&Z$→ 2×Random 3-spin Ising
Depolarising → Random interacting 8-vertex[2]



[1]Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2]Bombin et.al., PRX 2012, `doi:10/crz5`, `arXiv:1202.1852`

Phase diagram sketch



Consider the free energy cost of a logical error L (generalised domain wall),

$$\begin{aligned}\Delta_E(L) :=& F_{EL} - F_E \\ =& \frac{\log Z_E - \log Z_{EL}}{\beta}.\end{aligned}$$

Along the Nishimori line we have

$$\beta \Delta_E(L) = \log \frac{\Pr(\overline{E})}{\Pr(\overline{EL})},$$

which means

$$\begin{aligned}\text{Below th.}: &\quad \Delta_E(L) \to \infty \text{ (in mean)} \\ \text{Above th.}: &\quad \Delta_E(L) \to 0 \text{ (in prob.)}\end{aligned}$$

Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can also consider the following correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

$$\Pr(E) = \prod_j \phi_j\left(E_{R_j}\right)$$

This includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can also consider the following correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

$$\Pr(E) = \prod_j \phi_j\left(E_{R_j}\right)$$

This includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

Correlated case

By construction, we can extend to the correlated case by changing $\sigma \in \mathcal{P}_i$ to $\sigma \in \mathcal{P}_{R_j}$:

$$H_E(\vec{s}) := -\sum_j \sum_{\sigma \in \mathcal{P}_{R_j}} J_j(\sigma) \llbracket \sigma, E \rrbracket \prod_{k:\llbracket \sigma, S_k \rrbracket = -1} s_k$$

Nishimori condition: $$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket ,$$

As before we get that $Z_E = \Pr(\overline{E})$, and so the threshold manifests as a phase transition.

Correlated case

By construction, we can extend to the correlated case by changing $\sigma \in \mathcal{P}_i$ to $\sigma \in \mathcal{P}_{R_j}$:

$$H_E(\vec{s}) := -\sum_j \sum_{\sigma \in \mathcal{P}_{R_j}} J_j(\sigma) \, [\![\sigma, E]\!] \prod_{k : [\![\sigma, S_k]\!] = -1} s_k$$

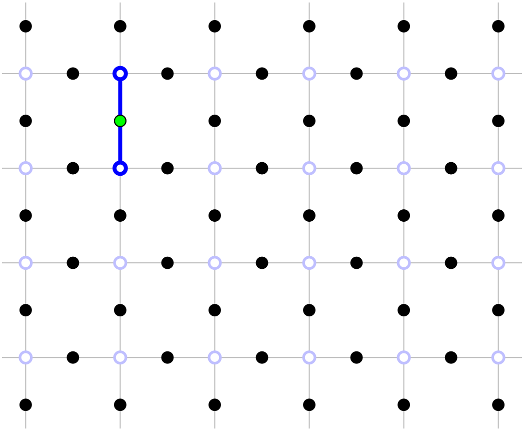
$$\text{Nishimori condition:} \qquad \beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \, [\![\sigma, \tau]\!] \,,$$

As before we get that $Z_E = \Pr(\overline{E})$, and so the threshold manifests as a phase transition.

Correlated example

Toric code with correlated bit-flips

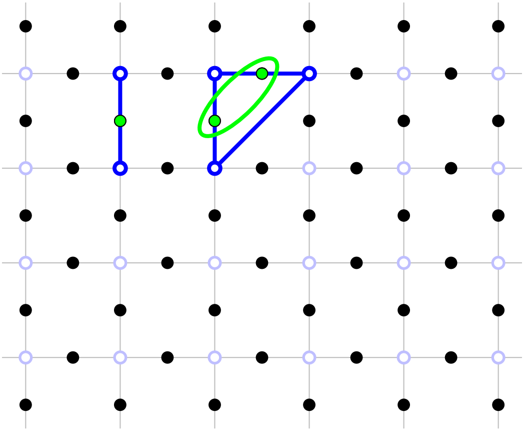
Correlations induce longer-range interactions



Correlated example

Toric code with correlated bit-flips

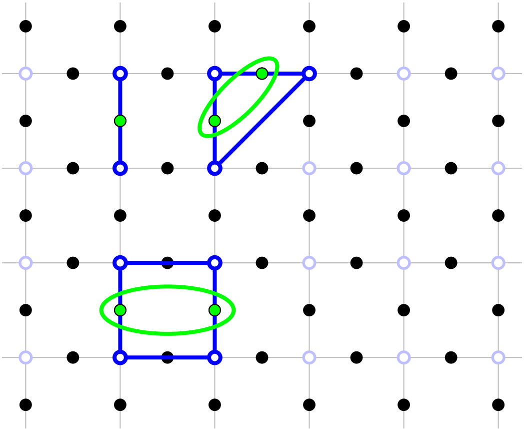
Correlations induce longer-range interactions



Correlated example

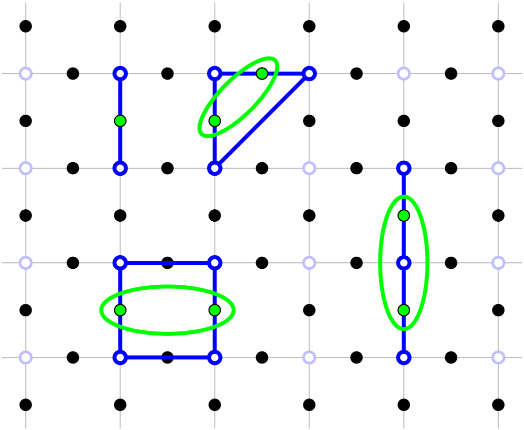
Toric code with correlated bit-flips

Correlations induce longer-range interactions

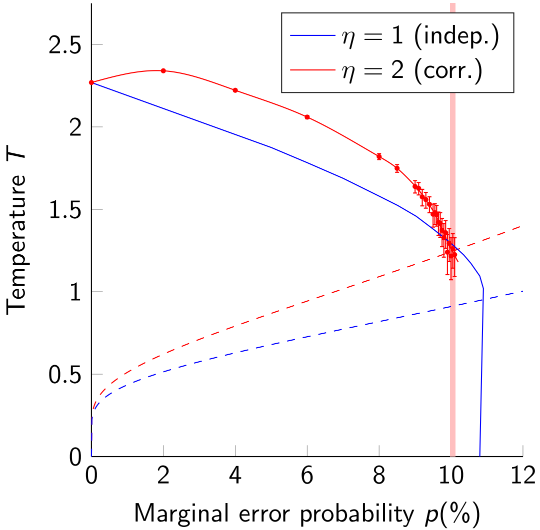


Correlated example

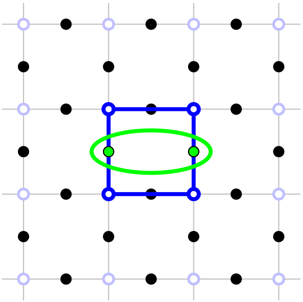
Toric code with correlated bit-flips
Correlations induce longer-range interactions



Monte Carlo simulations



$$\eta = \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}$$



Thresholds

Indep.: $p_{\mathrm{th}} = 10.917(3)\%$[1,2]

Corr.: $p_{\mathrm{th}} = 10.04(6)\%$

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`

[2] Toldin et.al., JSP 2009, `doi:10/c3r2kc`, `arXiv:0811.2101`

Other results

TN approximations of partition function give efficient approximate ML decoders (generalises BSV).

Other codes:

- Qudit subsystem codes
- Abelian quantum doubles (finite and infinite)
- General quantum doubles with pure-magnetic or pure-electric noise

Other noise models:

- Faulty measurements
- Leakage errors
- Spatio-temporal correlations, e.g. circuit noise

Other results

TN approximations of partition function give efficient approximate ML decoders (generalises BSV).

Other codes:

- Qudit subsystem codes
- Abelian quantum doubles (finite and infinite)
- General quantum doubles with pure-magnetic or pure-electric noise

Other noise models:

- Faulty measurements
- Leakage errors
- Spatio-temporal correlations, e.g. circuit noise

Other results

TN approximations of partition function give efficient approximate ML decoders (generalises BSV).

Other codes:

- Qudit subsystem codes
- Abelian quantum doubles (finite and infinite)
- General quantum doubles with pure-magnetic or pure-electric noise

Other noise models:

- Faulty measurements
- Leakage errors
- Spatio-temporal correlations, e.g. circuit noise

Conclusions and further work

- Extended the stat. mech. mapping to correlated models
- Can apply stat. mech. mapping to circuit noise via the history code
- Numerically evaluated the threshold of correlated bit-flips in the toric code
- Stat. mech. mapping gives tensor network maximum likelihood decoders

- Experimentally relevant correlated models?
- Non-stochastic noise? Non-Pauli noise? Coherent noise?
- Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work with David Tuckett and Benjamin Brown).

Thank you!

ArXiv:1809.10704

(me@)christopherchubb.com