Statistical mechanical models for quantum codes subject to correlated noise

Christopher Chubb (EQUS USyd), Steve Flammia (EQUS USyd/YQI Yale)

(me@)christopherchubb.com

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Université de Sherbrooke
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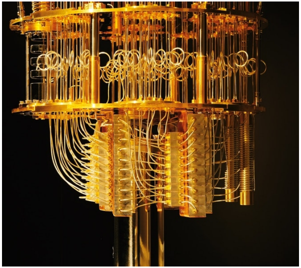
THE UNIVERSITY OF SYDNEY

EQUS

Quantum error correction

Quantum systems allow for powerful information processing...



Quantum computing (IBM Q)

Quantum communication/crypto (Micius)

...but are inherently vulernable to noise.

Quantum codes

Quantum codes allow us to suppress noise, making quantum information processing possible on realistic systems (threshold theorem).

Two important questions about quantum codes:

- How do I decode?
- What is the threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

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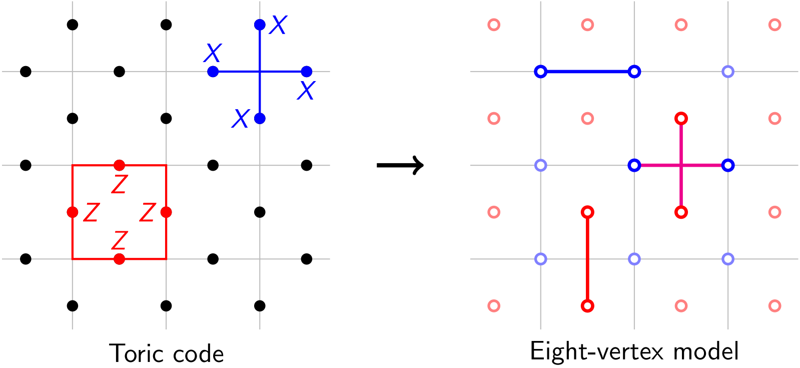
Statistical mechanical mapping

The idea here it to construct a family of statistical mechanical models, whose thermodynamic properties reflect the error correction properties of the code.



This will allow us to use the analytic and numerical tools developed to study stat mech systems to study quantum codes.

Statistical mechanical mapping

Stabiliser code & Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Threshold	$\longleftrightarrow$	Phase transition
Decoding	$\longleftrightarrow$	Calculating partition functions

Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:

Threshold approximation	$\longleftarrow$	Monte Carlo simulation
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Our results

- Generalise mapping to correlated noise for arbitrary codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

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Stabiliser codes and Pauli noise

For qubits, the Paulis $\mathcal{P} := \{I, X, Y, Z\}$ are defined

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will be considering stabiliser codes, which are specified by an Abelian subgroup of the Paulis $\mathcal{S}$, and whose code space $\mathcal{C}$ is the joint $+1$ eigenspace,

$$\mathcal{C} = \left\{ |\psi\rangle \, \middle| \, S\,|\psi\rangle = |\psi\rangle\,, \forall S \in \mathcal{S} \right\}.$$

Any two errors which differ by a stabiliser are logically equivalent, so the logical classes of errors are

$$\overline{E} := \{ES|S \in \mathcal{S}\}$$

Independent case: Hamiltonian

Let $[\![A, B]\!]$ be the scalar commutator of two Paulis, such that $AB =: [\![A, B]\!]\, BA$.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E, the (disordered) Hamiltonian H_E is defined

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k : [\![\sigma, S_k]\!] = -1} s_k}^{\text{DoF}}$$

for $s_k = \pm 1$, and coupling strengths $J_i(\sigma) \in \mathbb{R}$.

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis σ
- Disorder E flips some interactions (Ferro $\leftrightarrow$ Anti-ferro)
- Local code $\implies$ local stat. mech. model

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Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i),$$

we now want $Z_E = \Pr(\overline{E})$.

Using the gauge symmetry we have that the partition function can be written as a sum stabiliser-equivalent errors

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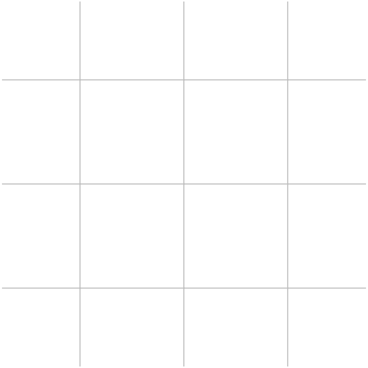
Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips



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$s_v = \pm 1$ on each vertex v

Step 2: Interactions

$$H_I = -\sum_{v\sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

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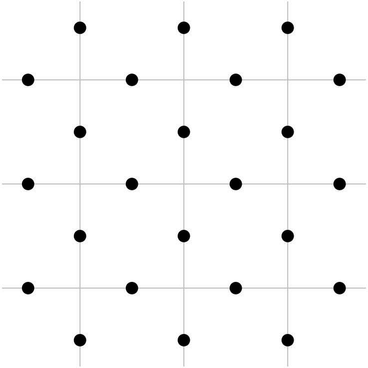

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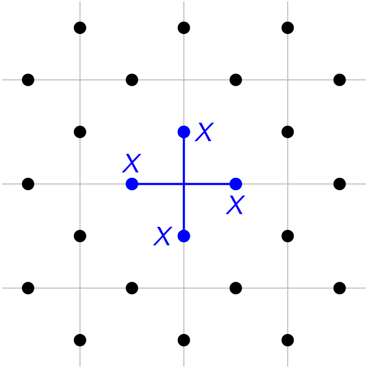

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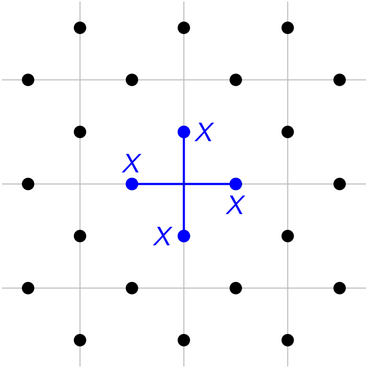

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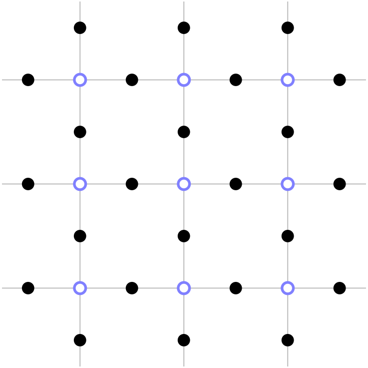

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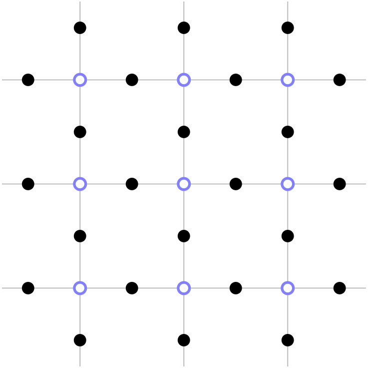

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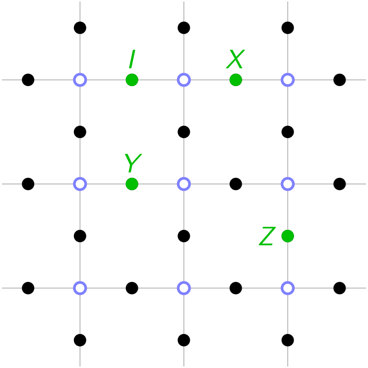

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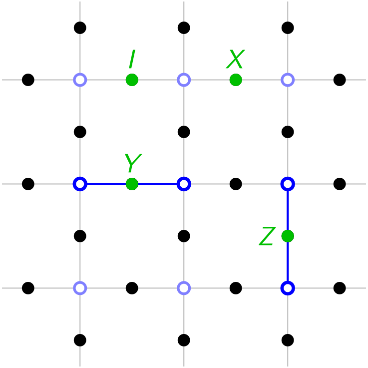

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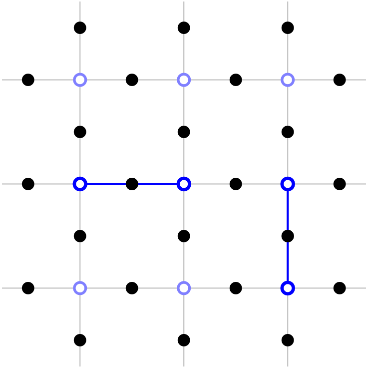

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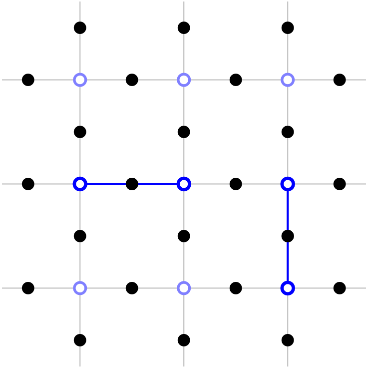

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Toric code and the random-bond Ising model

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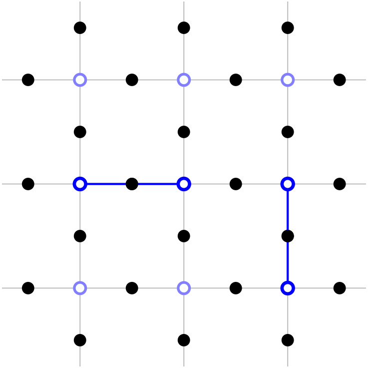

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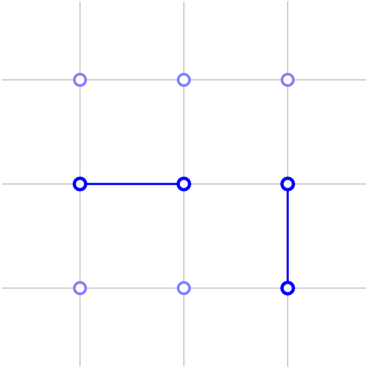
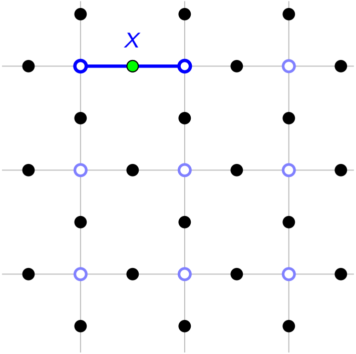


$\pm J$ Random-bond Ising Model

Other independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



Colour code

Bit-flip → Random 3-spin Ising
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Depolarising → Random interacting 8-vertex[2]

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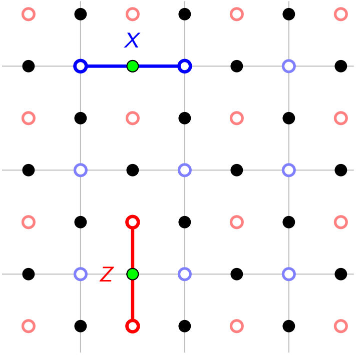

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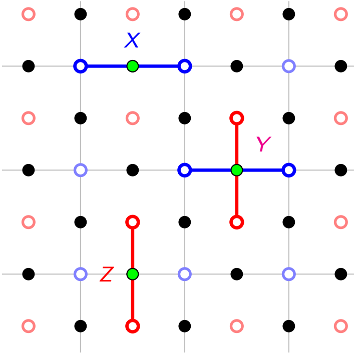

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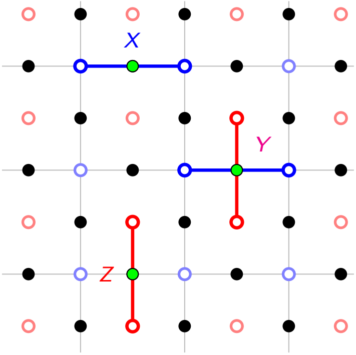

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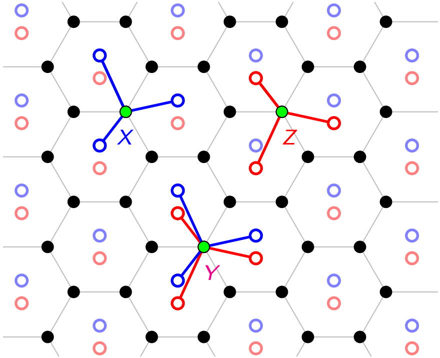

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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error L,

$$\Delta_E(L) = -\frac{1}{\beta}\log Z_{EL} + \frac{1}{\beta}\log Z_E .$$

Along the Nishimori line

$$\Delta_E(L) = \frac{1}{\beta}\log\frac{\Pr(\overline{E})}{\Pr(\overline{EL})},$$

which implies

$$\begin{aligned} \text{Below threshold :} \qquad & \Delta_E(L) \to \infty \ \text{(in mean)} \\ \text{Above threshold :} \qquad & \Delta_E(L) \to 0 \quad \text{(in prob.)} \end{aligned}$$

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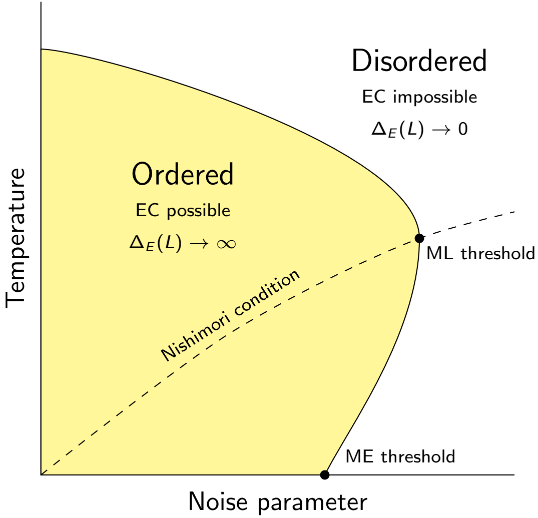
Phase diagram sketch



Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

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Nishimori condition: $$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket ,$$

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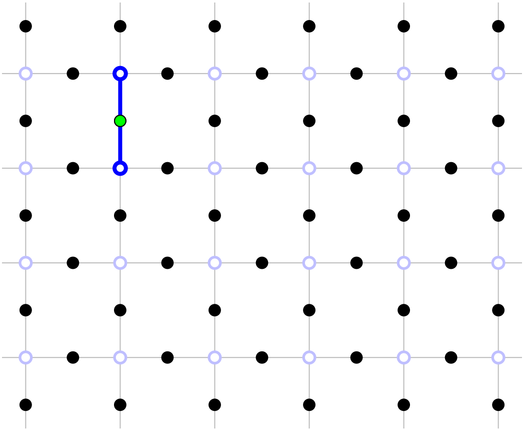
Correlated example

Toric code with correlated bit-flips
Correlations induce longer-range interactions



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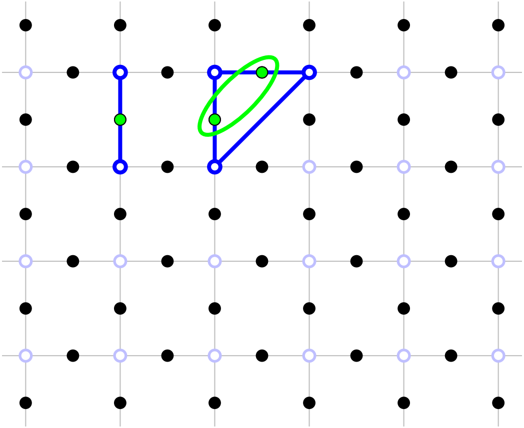

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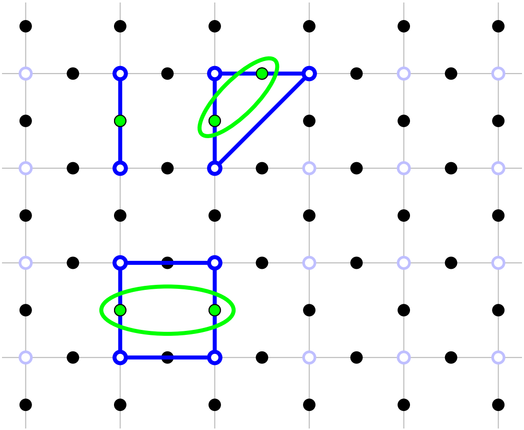

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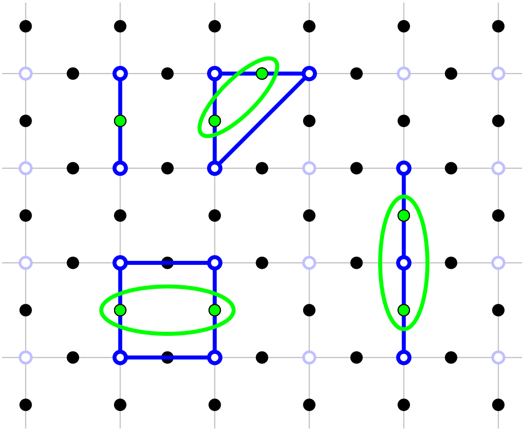
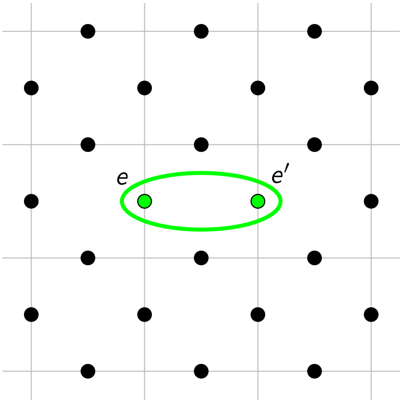


‘Across plaquette’ correlated bit-flips



This error model is entirely specified by the conditional error probabilities

$$\Pr(I_e|I_{e'}) \qquad \Pr(I_e|X_{e'})$$
$$\Pr(X_e|I_{e'}) \qquad \Pr(X_e|X_{e'})$$

for all neighbouring edges e and e'.

For our purposes, it will convenient to parameterise things by

$$p := \Pr(X_e), \qquad \eta := \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}.$$

Here p is the marginal error rate, and η is a measure of the correlations.

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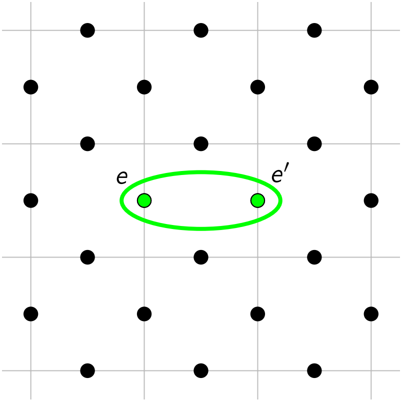

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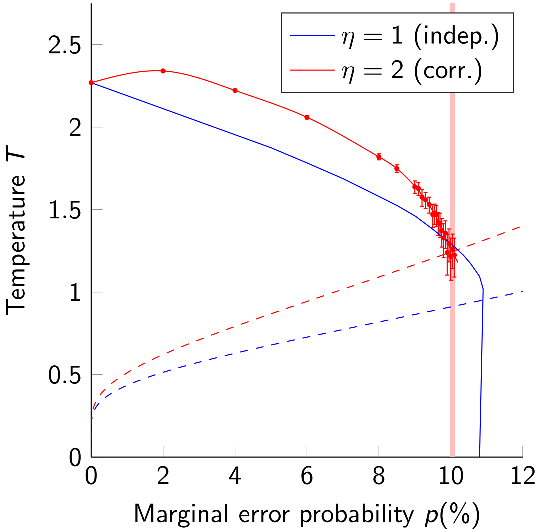
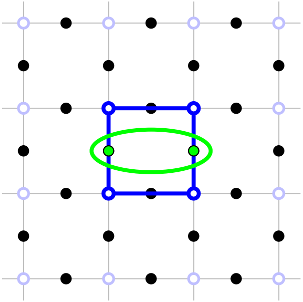
Monte Carlo simulations





Thresholds

Indep.: $p_{\mathrm{th}} = 10.917(2)\%$[1,2]
Corr.: $p_{\mathrm{th}} = 10.04(6)\%$

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2] Toldin et.al., JSP 2009, `doi:10/c3r2kc`, `arXiv:0811.2101`

Decoding

Can the stat. mech. model give us a decoder?

If an error E occurs, a decoder needs to select one of the degenerate logical error classes

$$\overline{E} \qquad \overline{EL_1} \qquad \overline{EL_2} \qquad \overline{EL_3} \quad \ldots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\mathrm{ML}} = \overline{EL_I} \qquad \text{where} \quad I = \arg\max_{l} \Pr\left(\overline{EL_l}\right).$$

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Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$l = \arg\max_{l} Z_{EL_l}.$$

Approximating Z_{EL_l} therefore allows us to approximate the ML decoder.

- Step 1: Measure the syndrome s
- Step 2: Construct an arbitrary error C_s which has syndrome s
- Step 3: Approximate $Z_{C_sL_l} = \Pr(\overline{C_sL_l})$ for each logical l
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Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks[2,3], allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo[4]:

[4] Verstraete et. al., PRL 2006, `doi:10/dfgcz8`, `arXiv:quant-ph/0601075`

[3] Bridgeman and Chubb, JPA 2017, `doi:10/cv7m`, `arXiv:1603.03039`

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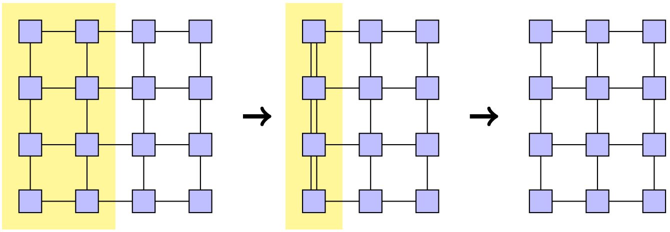

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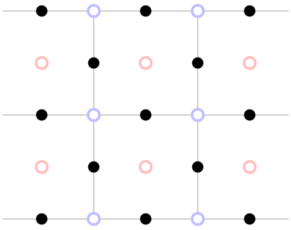
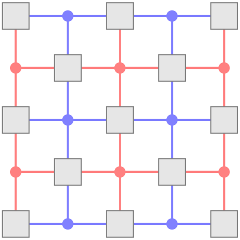
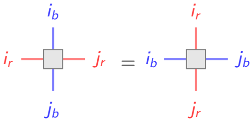
Decoding from (approximate) tensor network contraction

This gives an algorithm for (approximate) maximum likelihood decoding for any 2D code, subject to any locally correlated noise, generalising BSV.

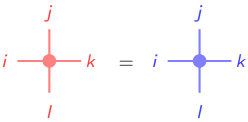
Indeed, applying this to iid noise in the surface code reproduces BSV:







$$= p\left(E_e \cdot X^{i_b+j_b} Z^{i_r+j_r}\right)$$



$$= \delta_{i,j,k,l}$$

Conclusions and further work

- Extended the stat. mech. mapping to correlated models
- Can apply stat. mech. mapping to circuit noise via the history code
- Numerically evaluated the threshold of correlated bit-flips in the toric code
- Stat. mech. mapping gives tensor network maximum likelihood decoders

- Can we apply this to experimentally relevant correlated models?
- Construction extends to all Abelian quantum doubles. Can we extend non-Abelian models?
- Non-Pauli noise? Coherent noise?
- Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work with David Tuckett and Benjamin Brown).

Thank you!

ArXiv:1809.10704

`(me@)christopherchubb.com`