Statistical mechanical models for stabiliser codes subject to correlated noise

Joint work with Steve Flammia (USyd/Yale)
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Quantum error correction

Two important questions about quantum codes:

- How do I decode?
- What is the threshold?

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Stabiliser code & iid Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Error-correcting threshold	$\longleftrightarrow$	Quenched phase transition
Maximum likelihood decoding	$\longleftrightarrow$	Calculating partition functions/free energies

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Our results

- Generalise mapping correlated noise for arbitrary codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code
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Independent case

Let $[\![A,B]\!]$ be the scalar commutator of two Paulis, $AB =: [\![A,B]\!]\,BA$.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E,

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma\in\mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma,E]\!]}^{\text{Disorder}} \overbrace{\prod_{k:[\![\sigma,S_k]\!]=-1} s_k}^{\text{DoF}}$$

for $s_k \in \{\pm 1\}$, and couplings $J_i(\sigma) \to \mathbb{R}$.

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis σ
- Disorder E flips some interactions (Ferro $\leftrightarrow$ Anti-ferro)
- Local code $\implies$ local stat. mech. model
- Stat. mech. model has a symmetry: $s_j \to -s_j$ and $E \to ES_j$

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Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i).$$

Nishimori condition: $$\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket ,$$

Using the Fourier-like orthogonality relation $\frac{1}{4} \sum_\sigma \llbracket \sigma, \tau \rrbracket = \delta_{\tau, I}$ we get that

$$e^{-\beta H_E(\vec{1})} = \Pr(E).$$

Together with the previous symmetry,

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_S \Pr(ES) = \Pr(\overline{E}).$$

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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical operator L,

$$\Delta_E(L) = -\frac{1}{\beta}\log Z_{EL} + \frac{1}{\beta}\log Z_E = \frac{1}{\beta}\log\frac{\Pr(\overline{E})}{\Pr(\overline{EL})}.$$

For a fixed error E

	Quantum code	Stat. mech. system
Below threshold	$\{\Pr(\overline{EL})\}_L$ peaked	$\Delta(L)\to\infty$ (in mean)
Above threshold	$\{\Pr(\overline{EL})\}_L$ flat	$\Delta(L)\to 0$ (in prob.)

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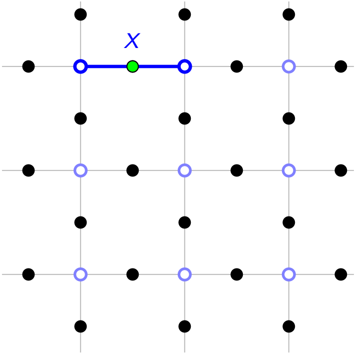
Independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



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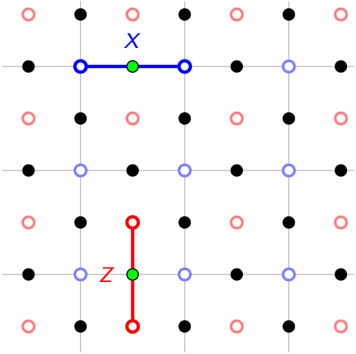

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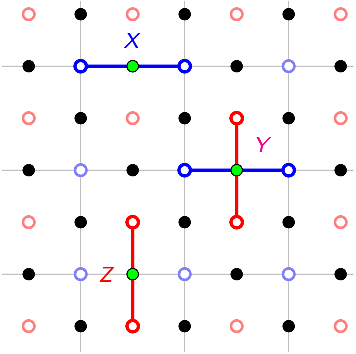

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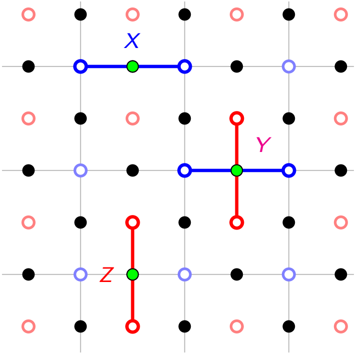

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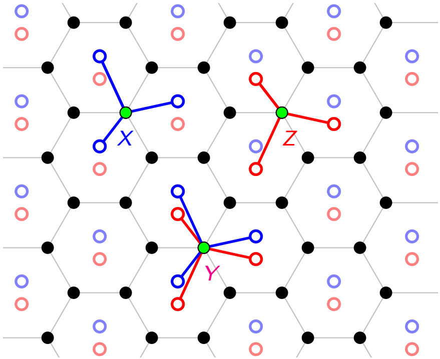

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Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

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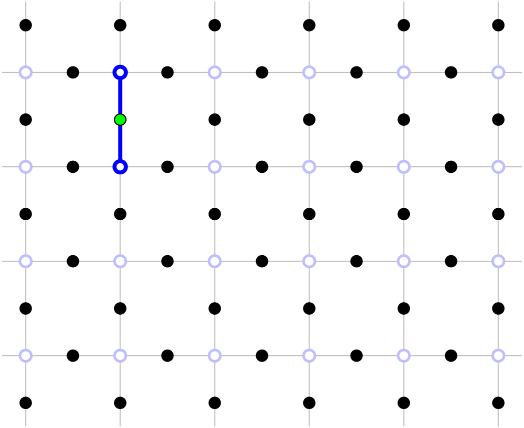
Toric code with correlated bit-flips
Correlations induce longer-range interactions



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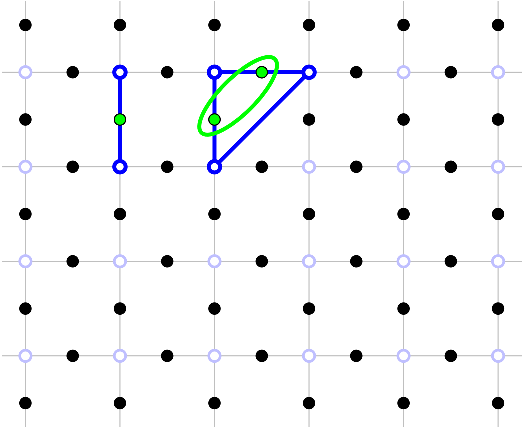

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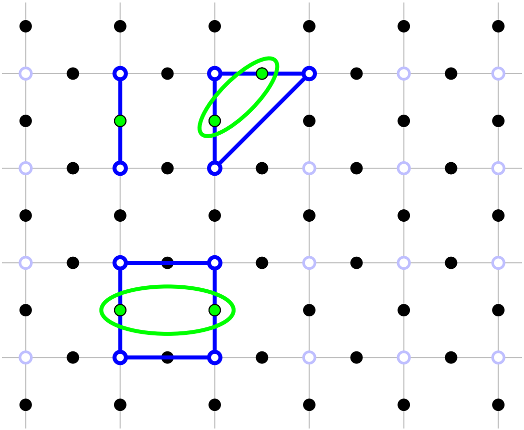

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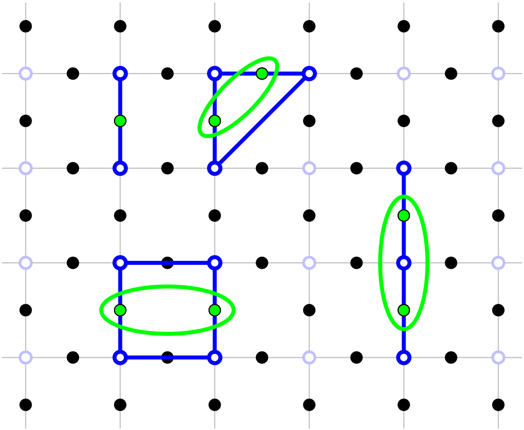
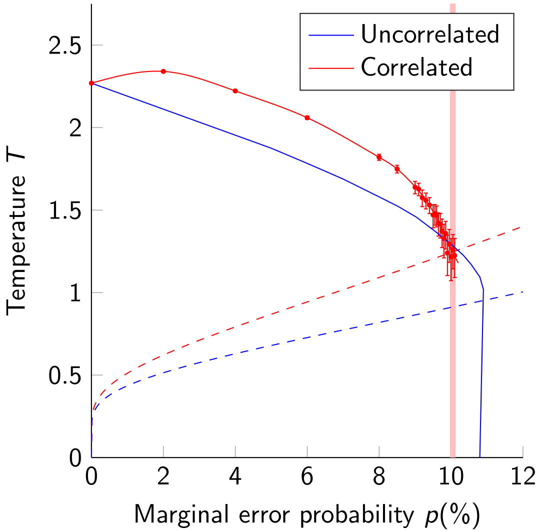
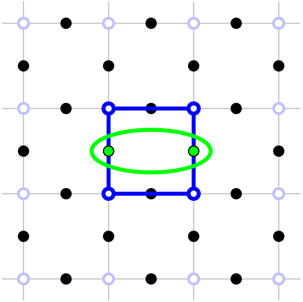


Monte Carlo simulations





Thresholds

Indep.[1]: $p_{\mathrm{th}} = 10.94(2)\%$
Corr.: $p_{\mathrm{th}} = 10.04(6)\%$

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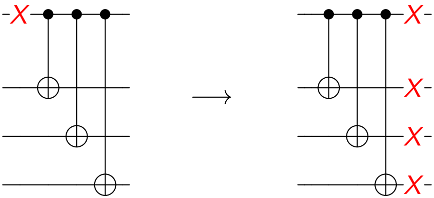
Circuit noise

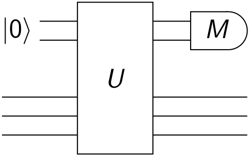
Noise followed by ideal measurements is unrealistic. In reality, circuits will be faulty.

Applying measurement circuits will tend to spread around and correlate noise:



Circuit noise

We will consider measurement circuits of the form



where U is a Clifford and M is a Pauli.

For convenience we only consider independent noise on each circuit. We also will push noise through until after the unitary:

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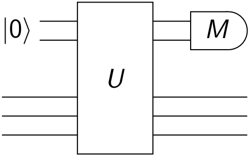

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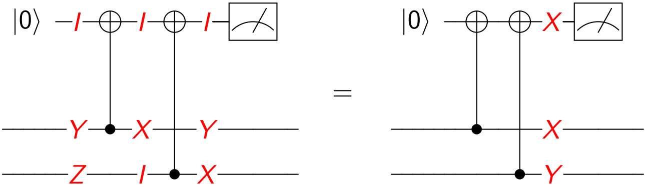

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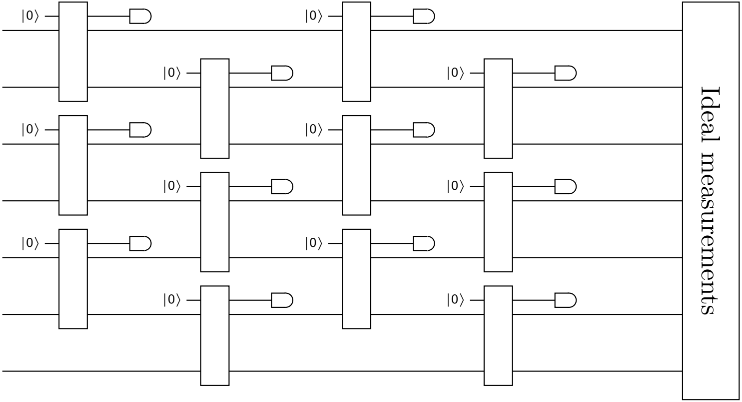

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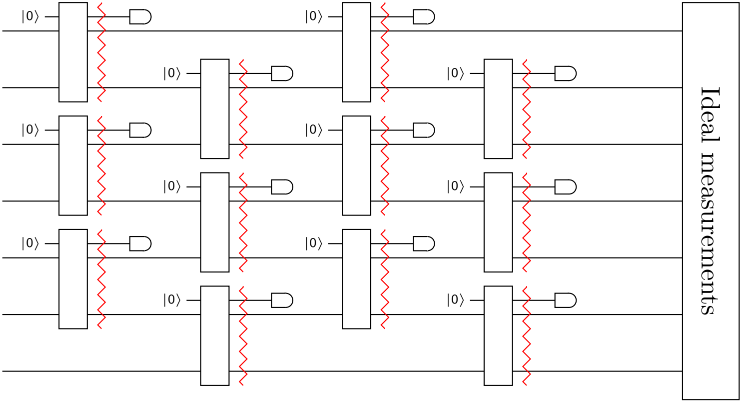

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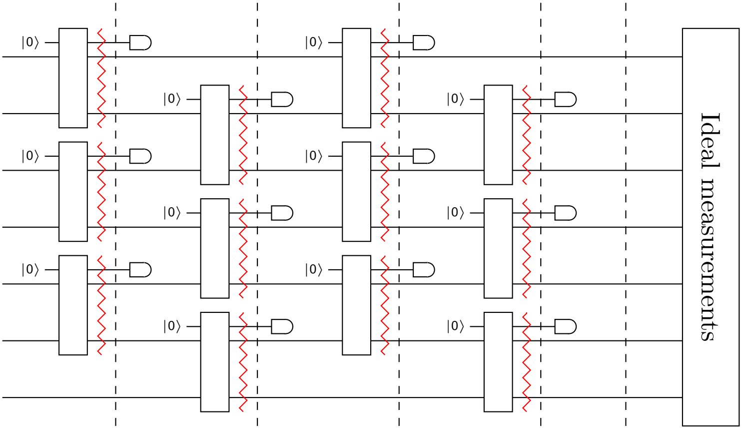

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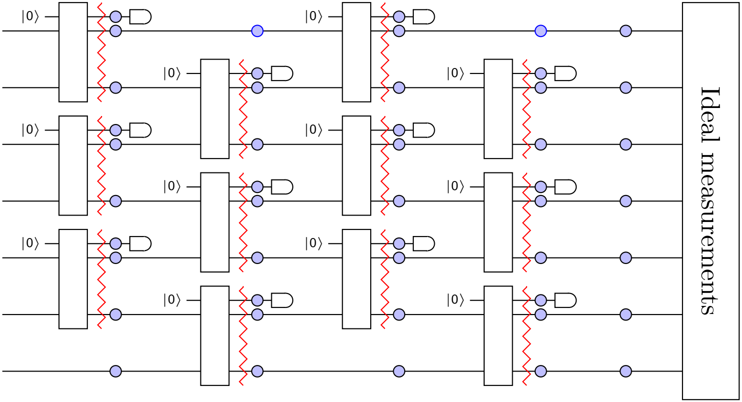

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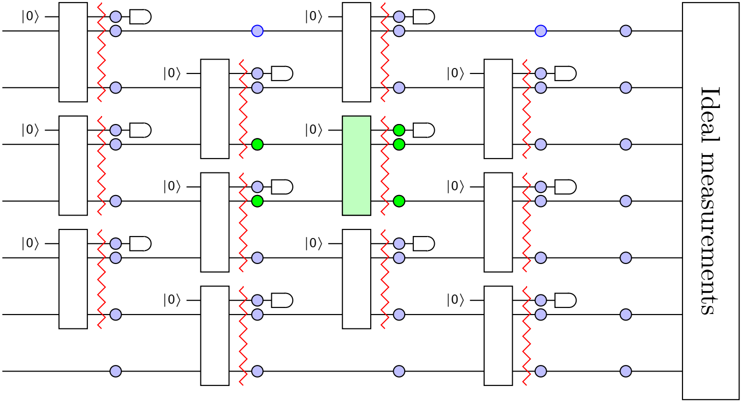

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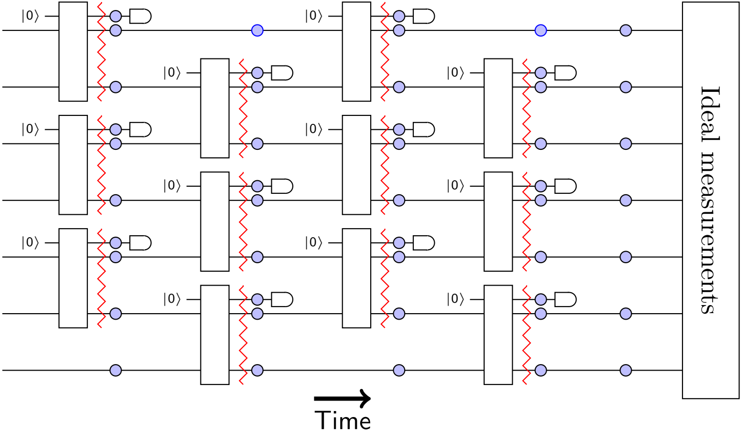

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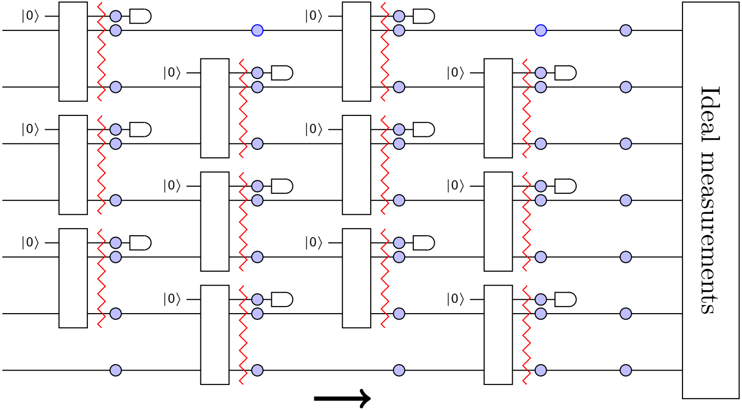

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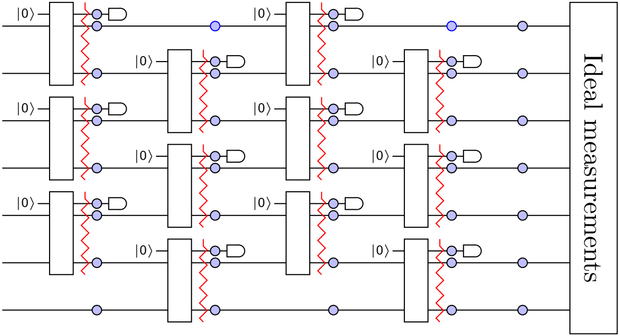
History code



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- Qubits placed at points in space-time (including ancillae)
- Stabiliers correspond to measurements and stabilisers at final time
- Logicals correspond to logicals at final time

- Circuit noise on original code = Spatially correlated noise on history code
- FT decoding of original code = Decoding the history code
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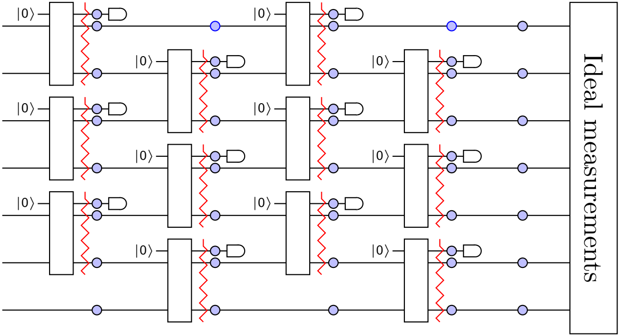

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Conclusions and further work

- Extended the stat. mech. mapping to correlated models
- Can apply stat. mech. mapping to circuit noise via the history code
- Stat. mech. mapping gives tensor network maximum likelihood decoders

- Can we apply this to experimentally relevant correlated models?
- Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work).

Thank you

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