Statistical mechanical models for quantum codes subject to correlated noise

Christopher Chubb (EQUS USyd), Steve Flammia (EQUS USyd/YQI Yale)

(me@)christopherchubb.com

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Quantum error correction

Two important questions about quantum codes:

- How do I decode?
- What is the threshold?

The statistical mechanical mapping[1] allows us to address both questions codes subject to Pauli noise.

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Statistical mechanical mapping

Stabiliser code & Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Error-correcting threshold	$\longleftrightarrow$	Quenched phase transition
Maximum likelihood decoding	$\longleftrightarrow$	Calculating partition functions/free energies

Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:

Threshold approximation	$\longleftarrow$	Monte Carlo simulation
Optimal decoding	$\longleftarrow$	Partition function calculation

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Our results

- Generalise mapping correlated noise for arbitrary codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

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Independent case

Let $[\![A, B]\!]$ be the scalar commutator of two Paulis $AB =: [\![A, B]\!]\, BA$, and restrict to qubit stabiliser codes.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E, the (disordered) Hamiltonian H_E is defined

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k:[\![\sigma, S_k]\!]=-1} s_k}^{\text{DoF}}$$

for $s_k = \pm 1$, and coupling strengths $J_i(\sigma) \in \mathbb{R}$.

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis σ
- Disorder E flips some interactions (Ferro $\leftrightarrow$ Anti-ferro)
- Local code $\implies$ local stat. mech. model
- Stat. mech. model has a symmetry: $s_k \to -s_k$ and $E \to ES_k$

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Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i).$$

Nishimori condition: $$\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \, [\![\sigma, \tau]\!]$$

Using the Fourier-like orthogonality relation $\frac{1}{4} \sum_\sigma [\![\sigma, \tau]\!] = \delta_{\tau, I}$ we get that

$$e^{-\beta H_E(\vec{1})} = \Pr(E).$$

Together with the previous symmetry,

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_S \Pr(ES) = \Pr(\overline{E}).$$

This intrinsically links the error correcting behaviour of the code to the thermodynamic behaviour of the model (along the Nishimori line).

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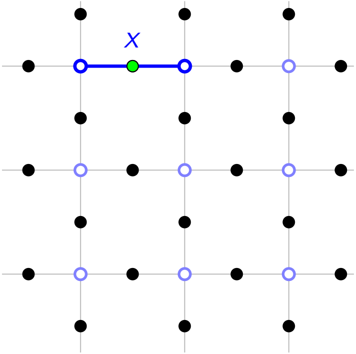
Independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



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Bit-flip → Random 3-spin Ising
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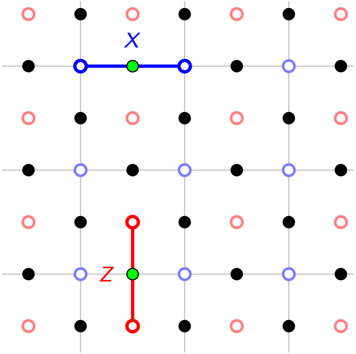

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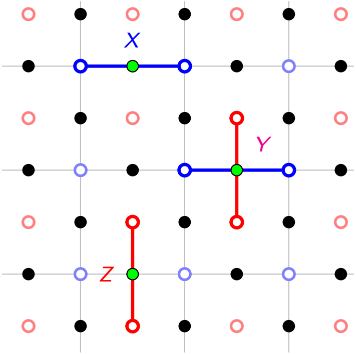

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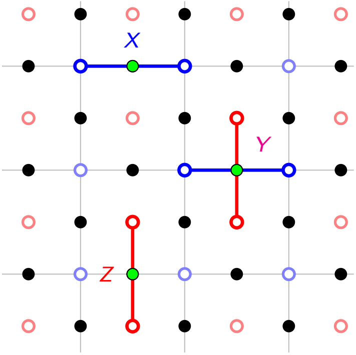

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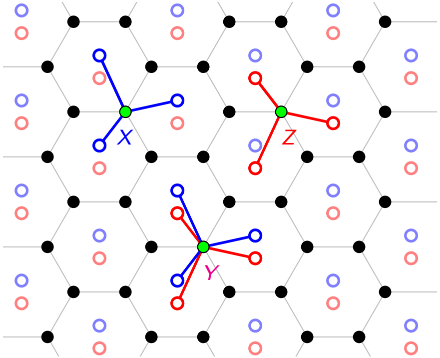

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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error L,

$$\Delta_E(L) = -\frac{1}{\beta}\log Z_{EL} + \frac{1}{\beta}\log Z_E.$$

Along the Nishimori line

$$\Delta_E(L) = \frac{1}{\beta}\log\frac{\Pr(\overline{E})}{\Pr(\overline{EL})},$$

which implies

$$\begin{aligned} \text{Below threshold :} \quad & \Delta_E(L) \to \infty \text{ (in mean)} \\ \text{Above threshold :} \quad & \Delta_E(L) \to 0 \quad \text{(in prob.)} \end{aligned}$$

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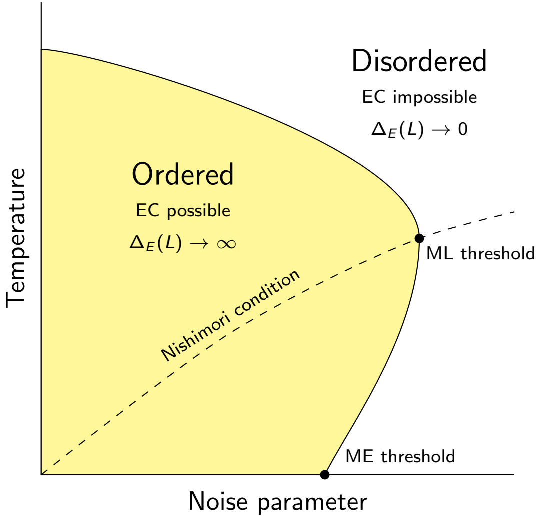
Phase diagram sketch



Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

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This model includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

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The construction works pretty much the same as before, changing $\sigma \in \mathcal{P}_i$ to $\sigma \in \mathcal{P}_{R_j}$:

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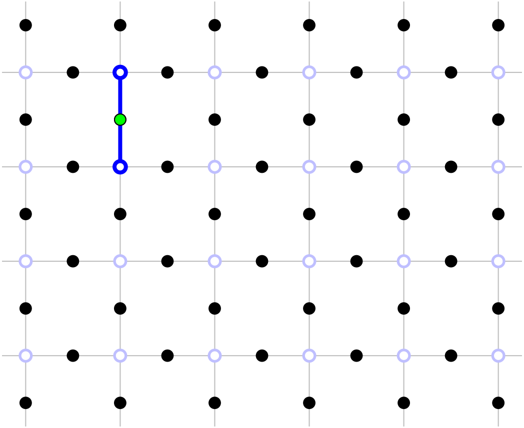
Toric code with correlated bit-flips

Correlations induce longer-range interactions



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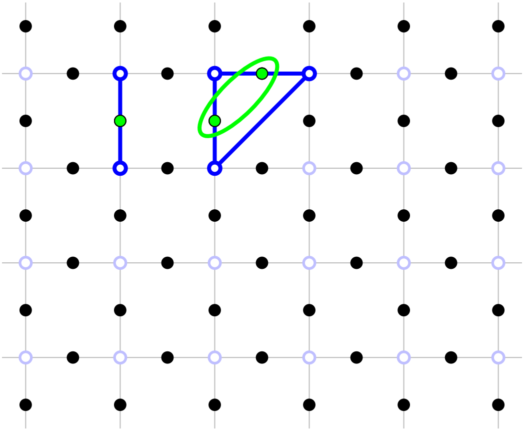

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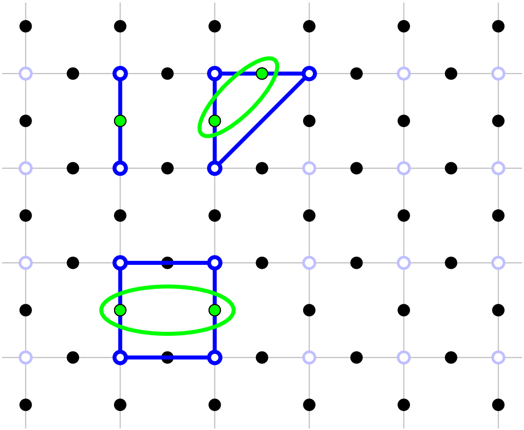

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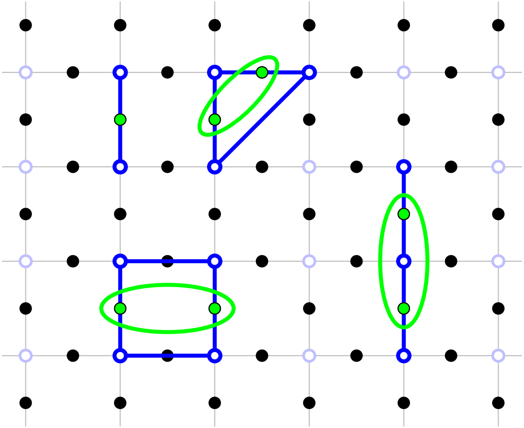
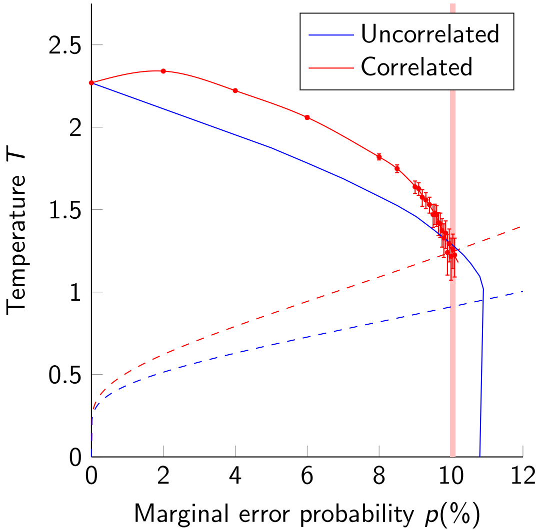
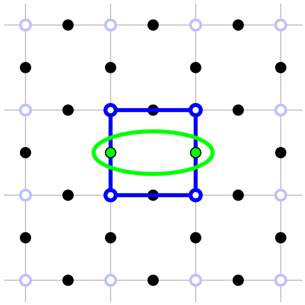


Monte Carlo simulations





Thresholds

Indep.[2,3]: $p_{\rm th} = 10.917(2)\%$
Corr.: $p_{\rm th} = 10.04(6)\%$

[3]Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[3]Toldin et.al., JSP 2009, `doi:10/c3r2kc`, `arXiv:0811.2101`

Decoding

Can the stat. mech. model give us a decoder?

If an error E occurs, a decoder needs to select one of the logical error classes

$$\overline{E} \qquad \overline{EL_1} \qquad \overline{EL_2} \qquad \overline{EL_3} \quad \ldots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\mathrm{ML}} = \overline{EL_l} \qquad \text{where} \quad l = \arg\max_l \Pr\left(\overline{EL_l}\right).$$

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Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$l = \arg\max_{l} Z_{EL_l}.$$

Approximating Z_{EL_l} therefore allows us to approximate the ML decoder.

- Step 1: Measure the syndrome s
- Step 2: Construct an error C_s which has syndrome s
- Step 3: Approximate $Z_{C_s L_l} = \Pr(\overline{C_s L_l})$ for each logical l
- Step 4: Find the l such that $Z_{C_s L_l}$ is maximised
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Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks[4,5], allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo[6]:

[4] Verstraete et. al., PRL 2006, doi:10/dfgcz8, arXiv:quant-ph/0601075
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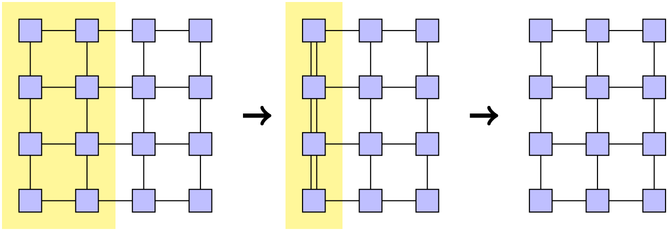

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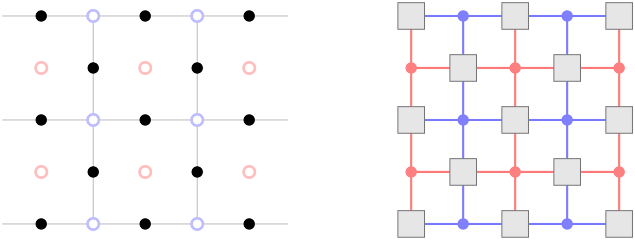
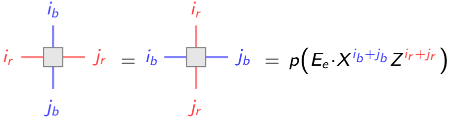
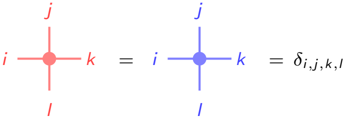
Decoding from (approximate) tensor network contraction

This gives an algorithm for (approximate) maximum likelihood decoding for any 2D code, subject to any locally correlated noise, generalising BSV.

Indeed, applying this to iid noise in the surface code reproduces BSV:







Conclusions and further work

- Extended the stat. mech. mapping to correlated models
- Can apply stat. mech. mapping to circuit noise via the history code
- Numerically evaluated the threshold of correlated bit-flips in the toric code
- Stat. mech. mapping gives tensor network maximum likelihood decoders

- Can we apply this to experimentally relevant correlated models?
- Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work with David Tuckett and Benjamin Brown).

Thank you!

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(me@)christopherchubb.com