

# Statistical mechanical models for quantum codes subject to correlated noise

Christopher Chubb (USyd), Steve Flammia (USyd/Yale)

(me@)christopherchubb.com

**arXiv:1809.10704**

Yale University  
2018-11



THE UNIVERSITY OF  
**SYDNEY**



# Quantum codes

Two important questions about quantum codes:

- How do I decode?
- What is the threshold?

The statistical mechanical mapping<sup>1</sup> allows us to address both questions codes subject to Pauli noise.

---

<sup>1</sup>Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

# Quantum codes

Two important questions about quantum codes:

- How do I decode **optimally**?
- What is the threshold?

The statistical mechanical mapping<sup>1</sup> allows us to address both questions codes subject to Pauli noise.

---

<sup>1</sup>Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

# Quantum codes

Two important questions about quantum codes:

- How do I decode optimally?
- What is the **fault-tolerant** threshold?

The statistical mechanical mapping<sup>1</sup> allows us to address both questions codes subject to Pauli noise.

---

<sup>1</sup>Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

# Quantum codes

Two important questions about quantum codes:

- How do I decode optimally?
- What is the fault-tolerant threshold?

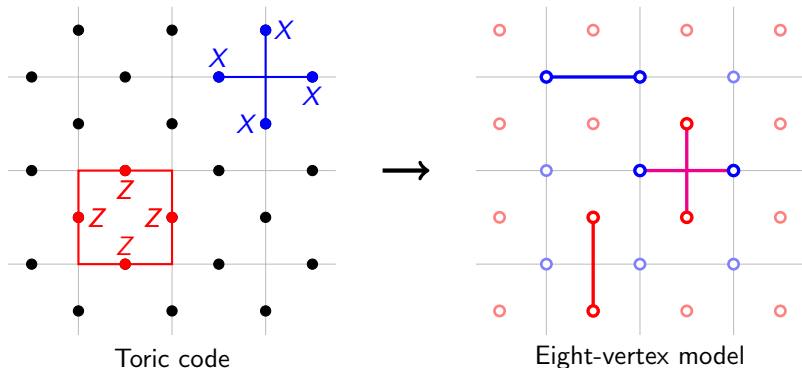
The statistical mechanical mapping<sup>1</sup> allows us to address both questions codes subject to Pauli noise.

---

<sup>1</sup>Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

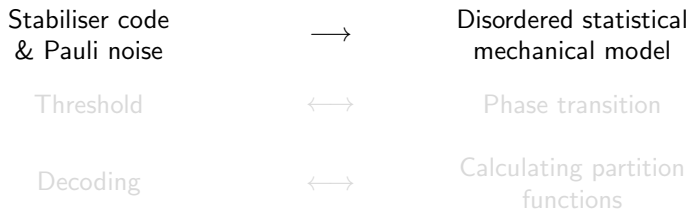
# Statistical mechanical mapping

The idea here is to construct a family of statistical mechanical models, whose thermodynamic properties reflect the error correction properties of the code.



This will allow us to use the analytic and numerical tools developed to study stat mech systems to study quantum codes.

# Statistical mechanical mapping



Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:



# Statistical mechanical mapping

Stabiliser code & Pauli noise	→	Disordered statistical mechanical model
Threshold	↔	Phase transition
Decoding	↔	Calculating partition functions

Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:

Threshold approximation	←	Monte Carlo simulation
Optimal decoding	←	Partition function calculation



# Statistical mechanical mapping

Stabiliser code & Pauli noise	→	Disordered statistical mechanical model
Threshold	↔	Phase transition
Decoding	↔	Calculating partition functions

Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:

Threshold approximation	←	Monte Carlo simulation
Optimal decoding	←	Partition function calculation

# Our results

- Generalise mapping to correlated noise for arbitrary subsystem codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

# Our results

- Generalise mapping to correlated noise for arbitrary subsystem codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

# Our results

- Generalise mapping to correlated noise for arbitrary subsystem codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

# Our results

- Generalise mapping to correlated noise for arbitrary subsystem codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

# Stabiliser codes and Pauli noise

For qubits, the Paulis  $\mathcal{P} := \{I, X, Y, Z\}$  are defined

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will be considering stabiliser codes, which are specified by an Abelian subgroup of the Paulis  $\mathcal{S}$ , and whose code space  $\mathcal{C}$  is the joint +1 eigenspace,

$$\mathcal{C} = \left\{ |\psi\rangle \mid \mathcal{S} |\psi\rangle = |\psi\rangle, \forall \mathcal{S} \in \mathcal{S} \right\}.$$

Any two errors which differ by a stabiliser are logically equivalent, so the logical classes of errors are

$$\bar{E} := \{ES \mid \mathcal{S} \in \mathcal{S}\}$$

# Independent case: Hamiltonian

Let  $\llbracket A, B \rrbracket$  be the scalar commutator of two Paulis, such that  $AB =: \llbracket A, B \rrbracket BA$ .

For a stabiliser code generated by  $\{S_k\}_k$ , and an error Pauli  $E$ , the (disordered) Hamiltonian  $H_E$  is defined

$$H_E(\vec{s}) := - \sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{\llbracket \sigma, E \rrbracket}^{\text{Disorder}} \overbrace{\prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k}^{\text{DoF}}$$

for  $s_k = \pm 1$ , and coupling strengths  $J_i(\sigma) \in \mathbb{R}$ .

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis  $\sigma$
- Disorder  $E$  flips some interactions (Ferro  $\leftrightarrow$  Anti-ferro)
- Local code  $\implies$  local stat. mech. model

# Independent case: Hamiltonian

Let  $\llbracket A, B \rrbracket$  be the scalar commutator of two Paulis, such that  $AB =: \llbracket A, B \rrbracket BA$ .

For a stabiliser code generated by  $\{S_k\}_k$ , and an error Pauli  $E$ , the (disordered) Hamiltonian  $H_E$  is defined

$$H_E(\vec{s}) := - \sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{\llbracket \sigma, E \rrbracket}^{\text{Disorder}} \overbrace{\prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k}^{\text{DoF}}$$

for  $s_k = \pm 1$ , and coupling strengths  $J_i(\sigma) \in \mathbb{R}$ .

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis  $\sigma$
- Disorder  $E$  flips some interactions (Ferro  $\leftrightarrow$  Anti-ferro)
- Local code  $\implies$  local stat. mech. model



# Independent case: Hamiltonian

Let  $\llbracket A, B \rrbracket$  be the scalar commutator of two Paulis, such that  $AB =: \llbracket A, B \rrbracket BA$ .

For a stabiliser code generated by  $\{S_k\}_k$ , and an error Pauli  $E$ , the (disordered) Hamiltonian  $H_E$  is defined

$$H_E(\vec{s}) := - \sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{\llbracket \sigma, E \rrbracket}^{\text{Disorder}} \overbrace{\prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k}^{\text{DoF}}$$

for  $s_k = \pm 1$ , and coupling strengths  $J_i(\sigma) \in \mathbb{R}$ .

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis  $\sigma$
- Disorder  $E$  flips some interactions (Ferro  $\leftrightarrow$  Anti-ferro)
- Local code  $\implies$  local stat. mech. model

## Independent case: Gauge symmetry

$$H_E(\vec{s}) = - \sum_i \sum_{\sigma \in \mathcal{P}_i} J_i(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

Using  $\llbracket A, B \rrbracket \llbracket A, C \rrbracket = \llbracket A, BC \rrbracket$ , we see this system has a gauge symmetry

$$s_k \rightarrow -s_k \quad \text{and} \quad E \rightarrow ES_k.$$

This gauge symmetry will capture the logical equivalence of errors,  $Z_E = Z_{ES_k}$ .

## Independent case: Gauge symmetry

$$H_E(\vec{s}) = - \sum_i \sum_{\sigma \in \mathcal{P}_i} J_i(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

Using  $\llbracket A, B \rrbracket \llbracket A, C \rrbracket = \llbracket A, BC \rrbracket$ , we see this system has a gauge symmetry

$$s_k \rightarrow -s_k \quad \text{and} \quad E \rightarrow ES_k.$$

This gauge symmetry will capture the logical equivalence of errors,  $Z_E = Z_{ES_k}$ .

# Independent case: Nishimori condition

Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i),$$

we now want  $Z_E = \Pr(\bar{E})$ .

Using the gauge symmetry we have that the partition function can be written as a sum stabiliser-equivalent errors

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_{F \in \bar{E}} e^{-\beta H_F(\vec{1})}.$$

If we select the coupling strength such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , then  $Z_E = \Pr(\bar{E})$  will follow.

# Independent case: Nishimori condition

Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i),$$

we now want  $Z_E = \Pr(\bar{E})$ .

Using the gauge symmetry we have that the partition function can be written as a sum stabiliser-equivalent errors

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_{F \in \bar{E}} e^{-\beta H_F(\vec{1})}.$$

If we select the coupling strength such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , then  $Z_E = \Pr(\bar{E})$  will follow.

# Independent case: Nishimori condition

Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i),$$

we now want  $Z_E = \Pr(\bar{E})$ .

Using the gauge symmetry we have that the partition function can be written as a sum stabiliser-equivalent errors

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_{F \in \bar{E}} e^{-\beta H_F(\vec{1})}.$$

If we select the coupling strength such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , then  $Z_E = \Pr(\bar{E})$  will follow.

## Independent case: Nishimori condition

We now want to pick our couplings such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ . Expanding this out, we get

$$\sum_i \log p_i(E) = - \sum_i \sum_{\sigma} \beta J_i(\sigma) \llbracket \sigma, E \rrbracket.$$

Using the Fourier-like orthogonality relation  $\frac{1}{4} \sum_{\sigma} \llbracket \sigma, \tau \rrbracket = \delta_{\tau, I}$ , this becomes

Nishimori condition: 
$$\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket,$$

which implies  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , and therefore  $Z_E = \Pr(\bar{E})$ .

This intrinsically links the error correcting behaviour of the code to the thermodynamic behaviour of the model (along the Nishimori line).

## Independent case: Nishimori condition

We now want to pick our couplings such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ . Expanding this out, we get

$$\sum_i \log p_i(E) = - \sum_i \sum_{\sigma} \beta J_i(\sigma) \llbracket \sigma, E \rrbracket.$$

Using the Fourier-like orthogonality relation  $\frac{1}{4} \sum_{\sigma} \llbracket \sigma, \tau \rrbracket = \delta_{\tau, I}$ , this becomes

Nishimori condition: 
$$\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket,$$

which implies  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , and therefore  $Z_E = \Pr(\bar{E})$ .

This intrinsically links the error correcting behaviour of the code to the thermodynamic behaviour of the model (along the Nishimori line).



## Independent case: Nishimori condition

We now want to pick our couplings such that  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ . Expanding this out, we get

$$\sum_i \log p_i(E) = - \sum_i \sum_{\sigma} \beta J_i(\sigma) \llbracket \sigma, E \rrbracket.$$

Using the Fourier-like orthogonality relation  $\frac{1}{4} \sum_{\sigma} \llbracket \sigma, \tau \rrbracket = \delta_{\tau, I}$ , this becomes

Nishimori condition: 
$$\beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) \llbracket \sigma, \tau \rrbracket,$$

which implies  $e^{-\beta H_E(\vec{1})} = \Pr(E)$ , and therefore  $Z_E = \Pr(\bar{E})$ .

This intrinsically links the error correcting behaviour of the code to the thermodynamic behaviour of the model (along the Nishimori line).

# Toric code and the random-bond Ising model

## Step 0: Code and noise model

### Toric code with iid bit-flips

#### Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

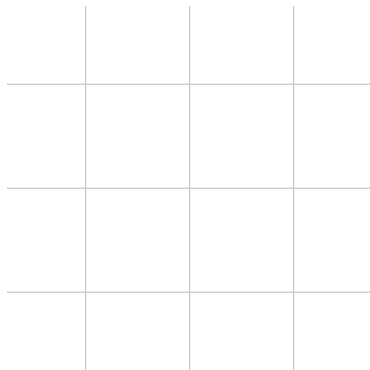
#### Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

#### Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

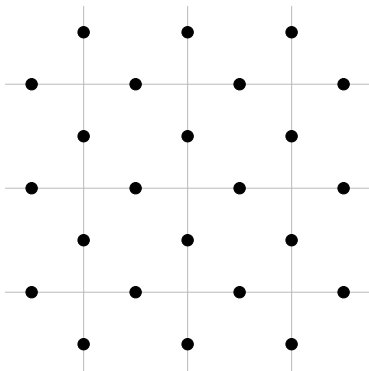
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

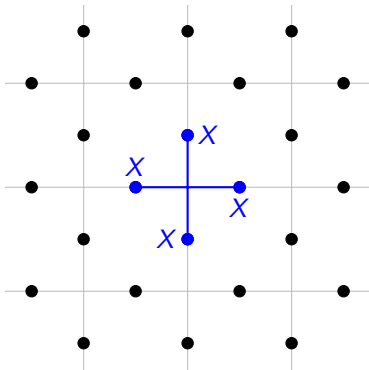
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

## Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

## Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

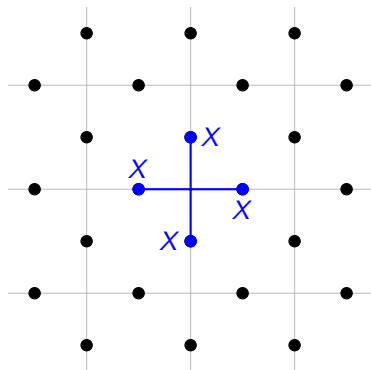
## Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

## Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

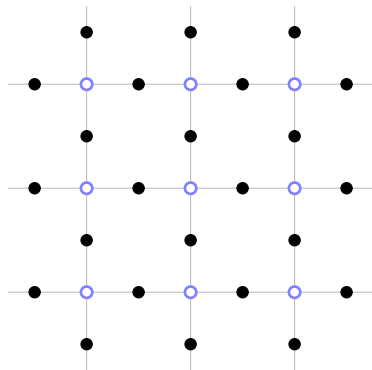
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

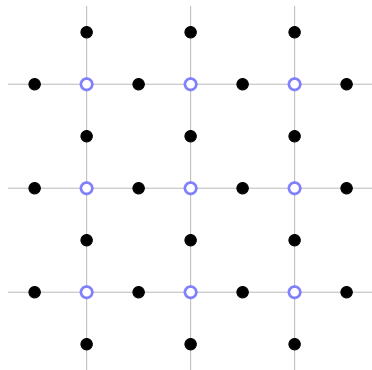
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

## Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

## Step 1: Degrees of freedom

$s_v = \pm 1$  on each vertex  $v$

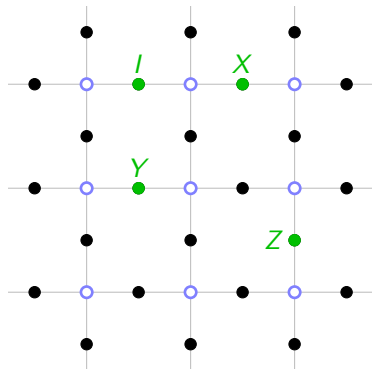
## Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

## Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$$[[\sigma_e, S_v]] = \begin{cases} -1 & \text{if } \sigma = X, Y, v \in e, \\ +1 & \text{else.} \end{cases}$$



# Toric code and the random-bond Ising model

## Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

## Step 1: Degrees of freedom

$s_v = \pm 1$  on each vertex  $v$

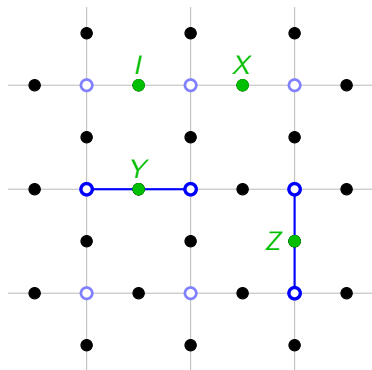
## Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J_{S_v S_{v'}}$$

## Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J_{e_{vv'}} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$$[[\sigma_e, S_v]] = \begin{cases} -1 & \text{if } \sigma = X, Y, v \in e, \\ +1 & \text{else.} \end{cases}$$

# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

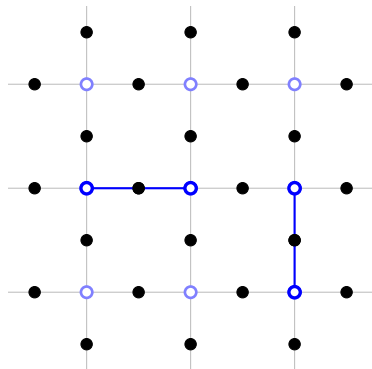
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

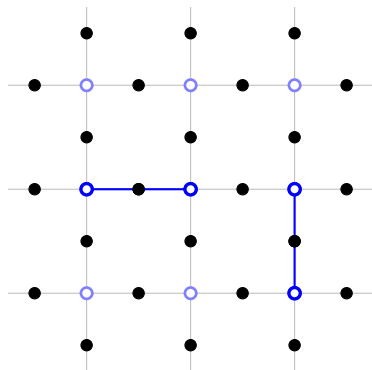
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$$[[\sigma, S_k]] \rightarrow [[\sigma, ES_k]]$$

# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

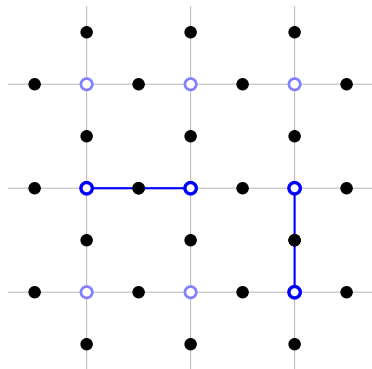
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$$[[\sigma, S_k]] \rightarrow [[\sigma, ES_k]]$$

# Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(+J) = p, \quad \Pr(-J) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

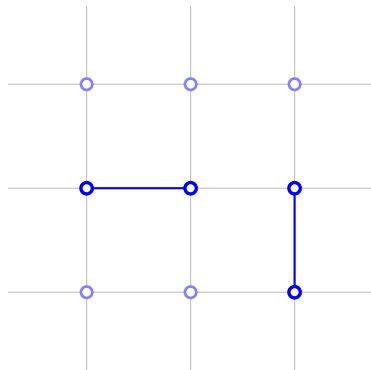
Step 2: Interactions

$$H_I = - \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$\pm J$  Random-bond Ising Model

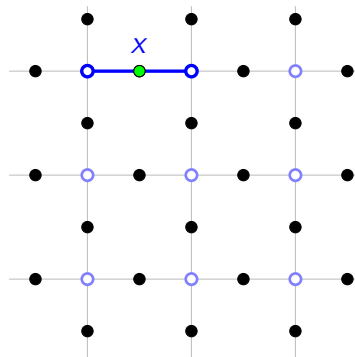
# Other independent examples

## Toric code

Bit-flip  $\rightarrow$  Random-bond Ising<sup>1</sup>

Indep.  $X$  &  $Z \rightarrow 2 \times$  Random-bond Ising

Depolarising  $\rightarrow$  Random 8-vertex model<sup>2</sup>

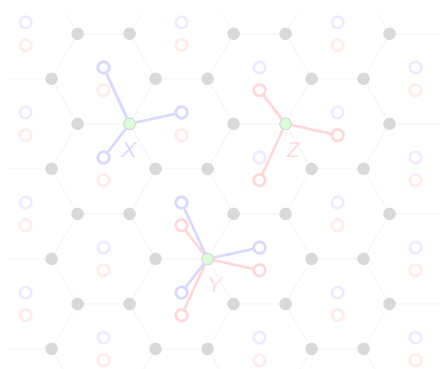


## Colour code

Bit-flip  $\rightarrow$  Random 3-spin Ising

Indep.  $X$  &  $Z \rightarrow 2 \times$  Random 3-spin Ising

Depolarising  $\rightarrow$  Random interacting 8-vertex<sup>2</sup>



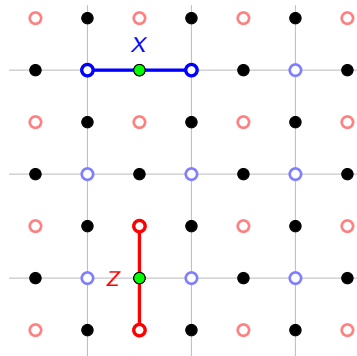
<sup>1</sup>Dennis et.al., JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

<sup>2</sup>Bombin et.al., PRX 2012, doi:10/crz5, arXiv:1202.1852

# Other independent examples

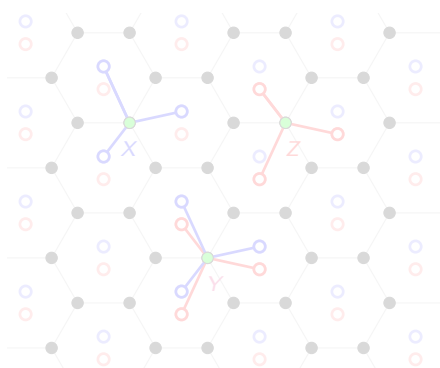
## Toric code

Bit-flip  $\rightarrow$  Random-bond Ising<sup>1</sup>  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random-bond Ising  
Depolarising  $\rightarrow$  Random 8-vertex model<sup>2</sup>



## Colour code

Bit-flip  $\rightarrow$  Random 3-spin Ising  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random 3-spin Ising  
Depolarising  $\rightarrow$  Random interacting 8-vertex<sup>2</sup>



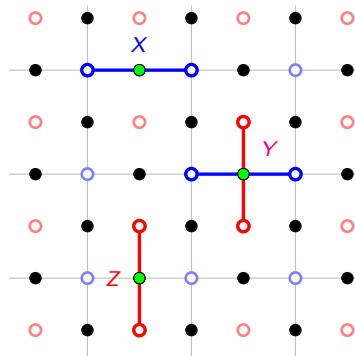
<sup>1</sup>Dennis et.al., JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

<sup>2</sup>Bombin et.al., PRX 2012, doi:10/crz5, arXiv:1202.1852

# Other independent examples

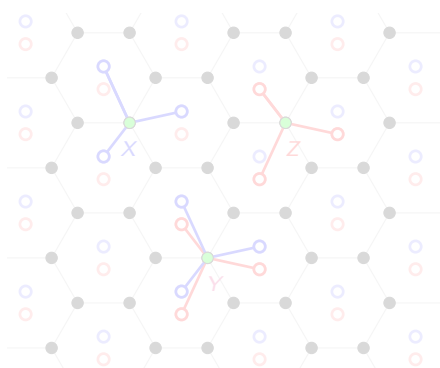
## Toric code

Bit-flip  $\rightarrow$  Random-bond Ising<sup>1</sup>  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random-bond Ising  
Depolarising  $\rightarrow$  Random 8-vertex model<sup>2</sup>



## Colour code

Bit-flip  $\rightarrow$  Random 3-spin Ising  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random 3-spin Ising  
Depolarising  $\rightarrow$  Random interacting 8-vertex<sup>2</sup>



<sup>1</sup>Dennis et.al., JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

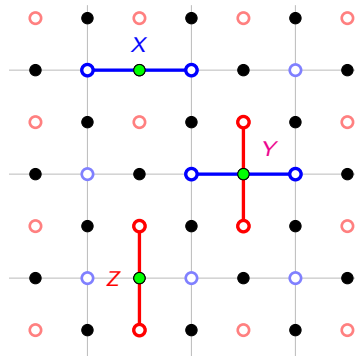
<sup>2</sup>Bombin et.al., PRX 2012, doi:10/crz5, arXiv:1202.1852



# Other independent examples

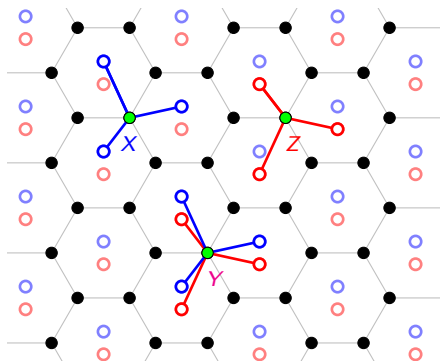
## Toric code

Bit-flip  $\rightarrow$  Random-bond Ising<sup>1</sup>  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random-bond Ising  
Depolarising  $\rightarrow$  Random 8-vertex model<sup>2</sup>



## Colour code

Bit-flip  $\rightarrow$  Random 3-spin Ising  
Indep.  $X$  &  $Z \rightarrow 2 \times$  Random 3-spin Ising  
Depolarising  $\rightarrow$  Random interacting 8-vertex<sup>2</sup>



<sup>1</sup>Dennis et.al., JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

<sup>2</sup>Bombin et.al., PRX 2012, doi:10/crz5, arXiv:1202.1852

# Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error  $L$ ,

$$\Delta_E(L) = -\frac{1}{\beta} \log Z_{EL} + \frac{1}{\beta} \log Z_E.$$

Along the Nishimori line

$$\Delta_E(L) = \frac{1}{\beta} \log \frac{\Pr(\bar{E})}{\Pr(EL)},$$

which implies

Below threshold :  $\Delta_E(L) \rightarrow \infty$  (in mean)

Above threshold :  $\Delta_E(L) \rightarrow 0$  (in prob.)

# Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error  $L$ ,

$$\Delta_E(L) = -\frac{1}{\beta} \log Z_{EL} + \frac{1}{\beta} \log Z_E.$$

Along the Nishimori line

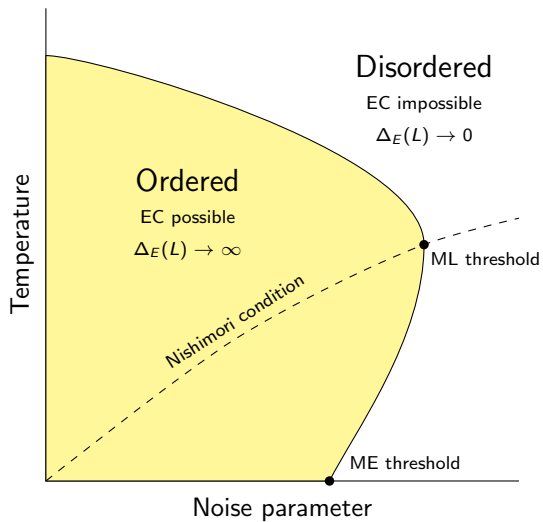
$$\Delta_E(L) = \frac{1}{\beta} \log \frac{\Pr(\bar{E})}{\Pr(EL)},$$

which implies

Below threshold :  $\Delta_E(L) \rightarrow \infty$  (in mean)

Above threshold :  $\Delta_E(L) \rightarrow 0$  (in prob.)

# Phase diagram sketch



# Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

## Factored distribution

An error model factors over regions  $\{R_j\}_j$  if there exist  $\phi_j : \mathcal{P}_{R_j} \rightarrow \mathbb{R}$  such that

$$\Pr(E) = \prod_j \phi_j(E_{R_j})$$

This model includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

# Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

## Factored distribution

An error model factors over regions  $\{R_j\}_j$  if there exist  $\phi_j : \mathcal{P}_{R_j} \rightarrow \mathbb{R}$  such that

$$\Pr(E) = \prod_j \phi_j(E_{R_j})$$

This model includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

# Correlated case

By construction, we can extend to the correlated case by changing  $\sigma \in \mathcal{P}_i$  to  $\sigma \in \mathcal{P}_{R_j}$ :

$$H_E(\vec{s}) := - \sum_j \sum_{\sigma \in \mathcal{P}_{R_j}} J_j(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

Nishimori condition: 
$$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket,$$

As before we get that  $Z_E = \Pr(\bar{E})$ , and so the threshold manifests as a phase transition.

# Correlated case

By construction, we can extend to the correlated case by changing  $\sigma \in \mathcal{P}_i$  to  $\sigma \in \mathcal{P}_{R_j}$ :

$$H_E(\vec{s}) := - \sum_j \sum_{\sigma \in \mathcal{P}_{R_j}} J_j(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

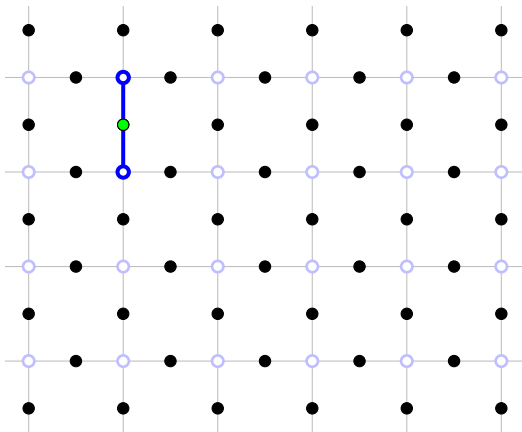
Nishimori condition: 
$$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket,$$

As before we get that  $Z_E = \Pr(\bar{E})$ , and so the threshold manifests as a phase transition.



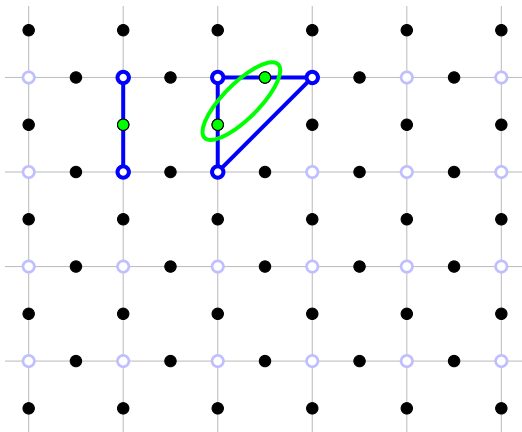
# Correlated example

**Toric code with correlated bit-flips**  
Correlations induce longer-range interactions



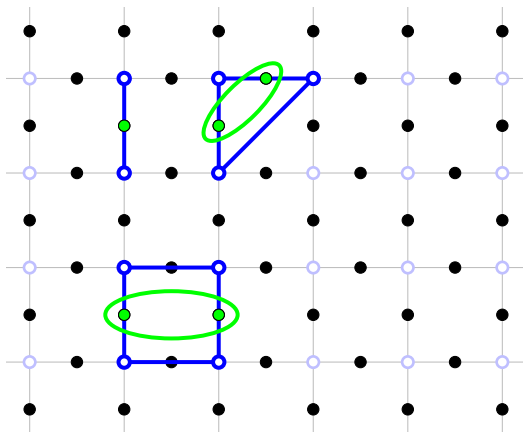
# Correlated example

**Toric code with correlated bit-flips**  
Correlations induce longer-range interactions



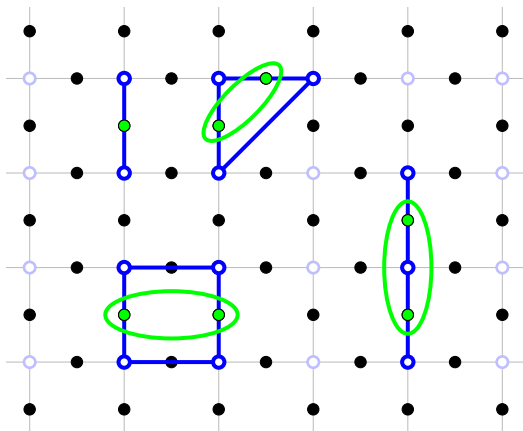
# Correlated example

**Toric code with correlated bit-flips**  
Correlations induce longer-range interactions

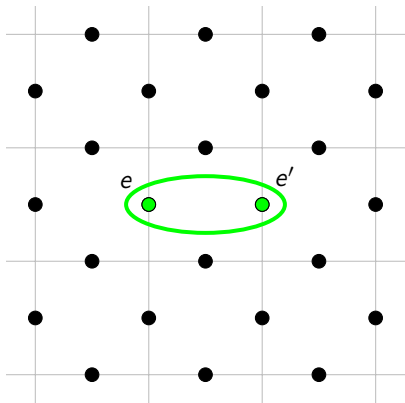


# Correlated example

**Toric code with correlated bit-flips**  
Correlations induce longer-range interactions



# 'Across plaquette' correlated bit-flips



This error model is entirely specified by the conditional error probabilities

$$\begin{array}{ll} \Pr(I_e|I_{e'}) & \Pr(I_e|X_{e'}) \\ \Pr(X_e|I_{e'}) & \Pr(X_e|X_{e'}) \end{array}$$

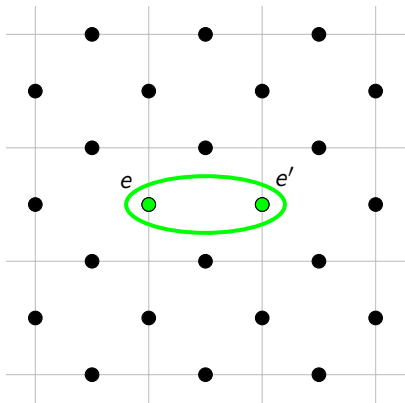
for all neighbouring edges  $e$  and  $e'$ .

For our purposes, it will be convenient to parameterise things by

$$p := \Pr(X_e), \quad \eta := \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}.$$

Here  $p$  is the marginal error rate, and  $\eta$  is a measure of the correlations.

# 'Across plaquette' correlated bit-flips



This error model is entirely specified by the conditional error probabilities

$$\begin{array}{ll} \Pr(I_e|I_{e'}) & \Pr(I_e|X_{e'}) \\ \Pr(X_e|I_{e'}) & \Pr(X_e|X_{e'}) \end{array}$$

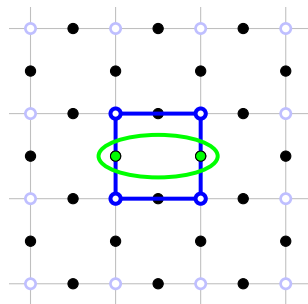
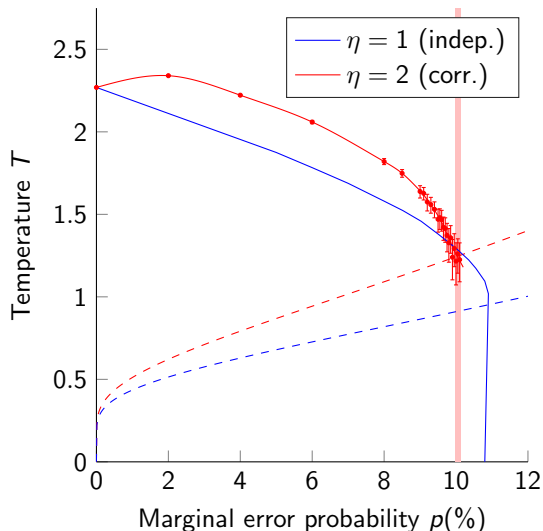
for all neighbouring edges  $e$  and  $e'$ .

For our purposes, it will be convenient to parameterise things by

$$p := \Pr(X_e), \quad \eta := \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}.$$

Here  $p$  is the marginal error rate, and  $\eta$  is a measure of the correlations.

# Monte Carlo simulations



## Thresholds

Indep.:  $p_{th} = 10.917(2)\%^{1,2}$

Corr.:  $p_{th} = 10.04(6)\%$

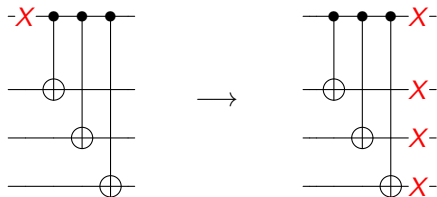
<sup>1</sup>Dennis et.al., JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

<sup>2</sup>Toldin et.al., JSP 2009, doi:10/c3r2kc, arXiv:0811.2101

# Circuit noise

Noise followed by ideal measurements is unrealistic. In reality, circuits will be faulty.

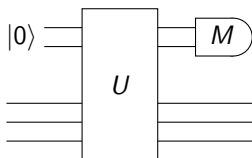
Noisy circuits give faulty outcomes, and will spread around and correlate noise:





# Circuit noise

We will consider measurement circuits of the form



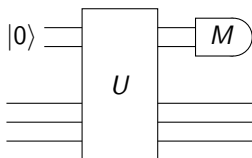
where  $U$  is a Clifford and  $M$  is a Pauli.

For convenience we only consider independent noise on each circuit. We also will push noise through until after the unitary:



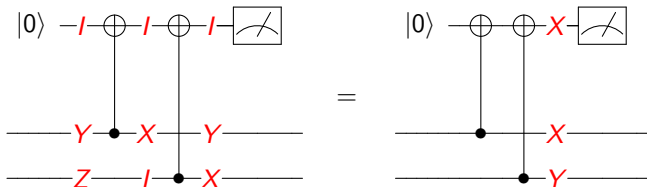
# Circuit noise

We will consider measurement circuits of the form

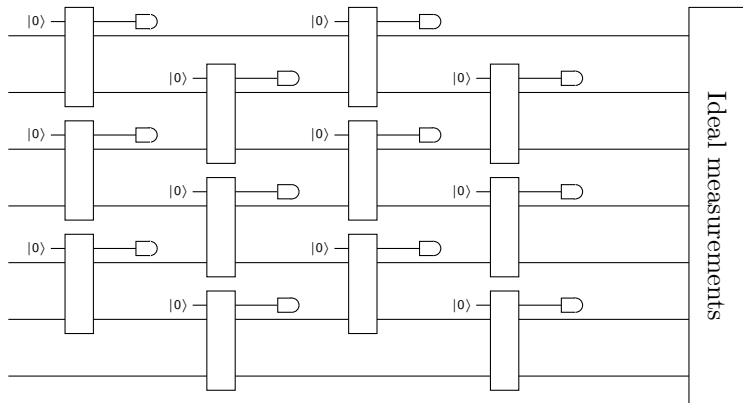


where  $U$  is a Clifford and  $M$  is a Pauli.

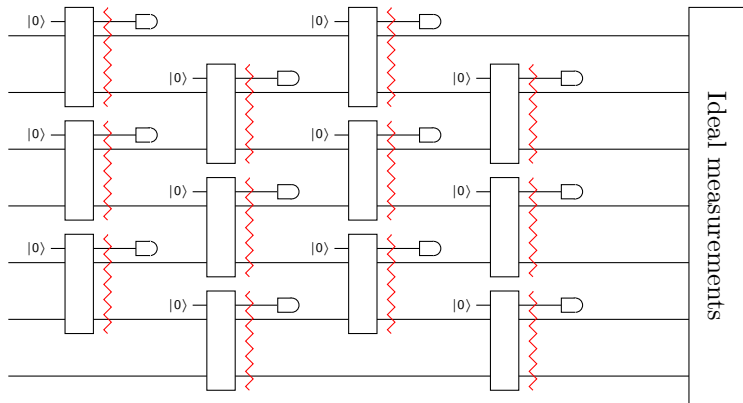
For convenience we only consider independent noise on each circuit. We also will push noise through until after the unitary:



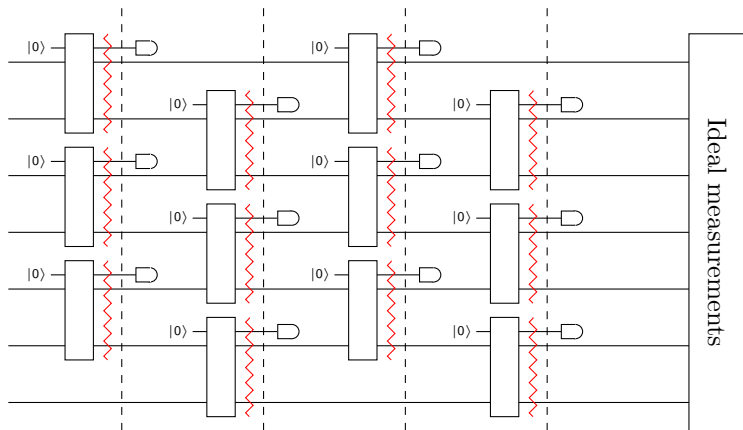
# Circuit noise



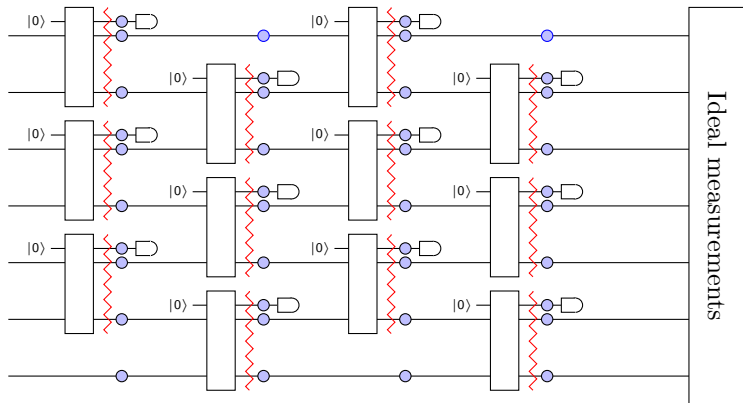
# Circuit noise



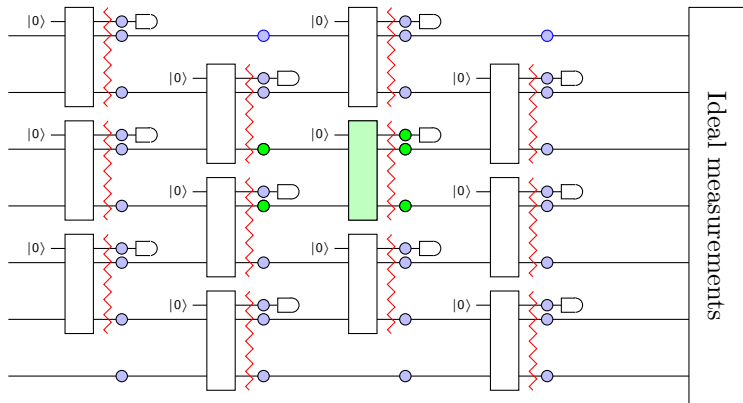
# Circuit noise



# Circuit noise

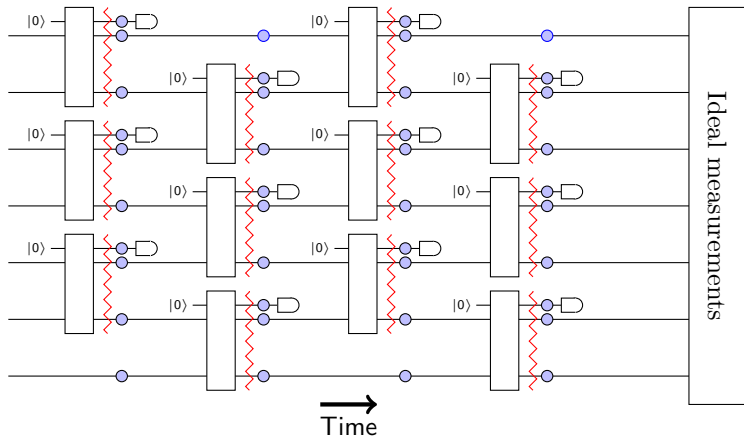


# Circuit noise



$$\Pr(\text{Error history}) = \prod_{j,t} \Pr(\text{Error after meas. } (x, t) \mid \text{Error before meas. } (x, t))$$

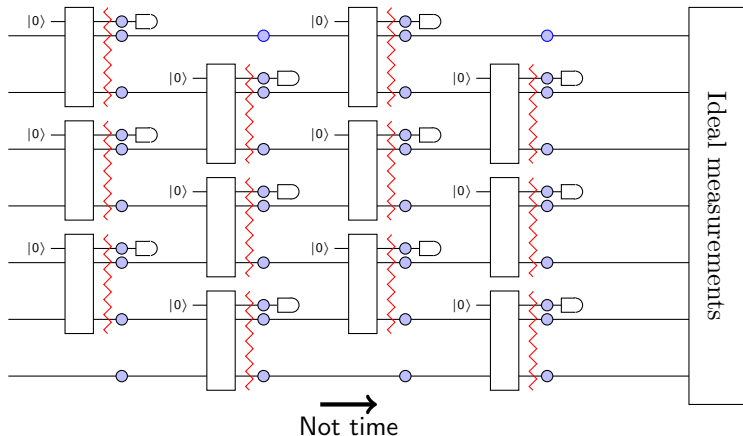
# Circuit noise



$$\Pr(\text{Error history}) = \prod_{j,t} \Pr(\text{Error after meas. } (x, t) \mid \text{Error before meas. } (x, t))$$

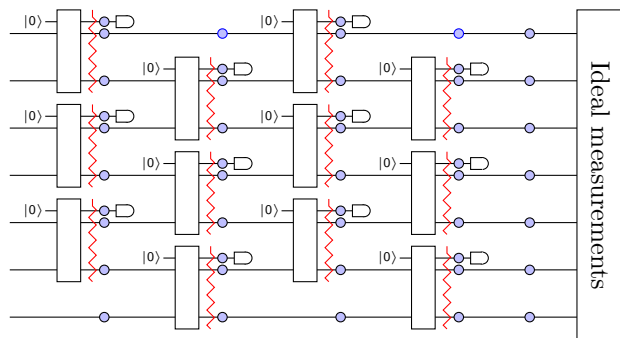


# Circuit noise



$$\Pr(\text{Error history}) = \prod_{j,t} \Pr(\text{Error after meas. } (x, t) \mid \text{Error before meas. } (x, t))$$

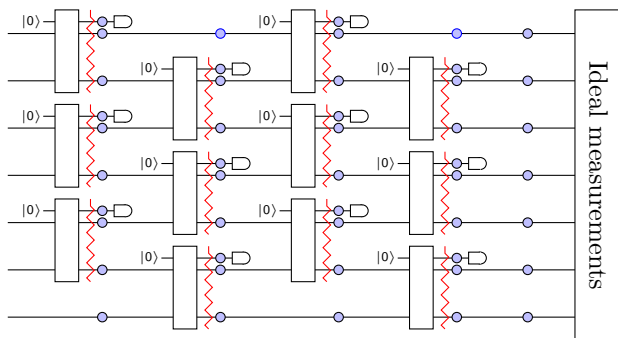
# History code



History code:

- Qubits placed at points in space-time (including ancillae)
  - Stabilisers correspond to measurements and stabilisers at final time
  - Logicals correspond to logicals at final time
- 
- Circuit noise on original code = Spatially correlated noise on history code
  - FT decoding of original code = Decoding the history code
  - FT threshold of original code = Threshold of history code

# History code



History code:

- Qubits placed at points in space-time (including ancillae)
- Stabilizers correspond to measurements and stabilisers at final time
- Logicals correspond to logicals at final time
  
- Circuit noise on original code = Spatially correlated noise on history code
- FT decoding of original code = Decoding the history code
- FT threshold of original code = Threshold of history code

# Decoding

Can the stat. mech. model give us a decoder?

If an error  $E$  occurs, a decoder needs to select one of the degenerate logical error classes

$$\overline{E} \quad \overline{EL_1} \quad \overline{EL_2} \quad \overline{EL_3} \quad \dots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\text{ML}} = \overline{EL_I} \quad \text{where} \quad I = \arg \max_l \Pr(\overline{EL_l}).$$

# Decoding

Can the stat. mech. model give us a decoder?

If an error  $E$  occurs, a decoder needs to select one of the degenerate logical error classes

$$\overline{E} \quad \overline{EL_1} \quad \overline{EL_2} \quad \overline{EL_3} \quad \dots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\text{ML}} = \overline{EL_I} \quad \text{where} \quad I = \arg \max_l \Pr(\overline{EL_l}).$$

# Decoding

Can the stat. mech. model give us a decoder?

If an error  $E$  occurs, a decoder needs to select one of the degenerate logical error classes

$$\overline{E} \quad \overline{EL_1} \quad \overline{EL_2} \quad \overline{EL_3} \quad \dots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\text{ML}} = \overline{EL_I} \quad \text{where} \quad I = \arg \max_I \Pr(\overline{EL_I}).$$

# Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$l = \arg \max_l Z_{EL_l}.$$

Approximating  $Z_{EL_l}$  therefore allows us to approximate the ML decoder.

- Step 1: Measure the syndrome  $s$
- Step 2: Construct an arbitrary error  $C_s$  which has syndrome  $s$
- Step 3: Approximate  $Z_{C_s L_l} = \Pr(\overline{C_s L_l})$  for each logical  $l$
- Step 4: Find the  $l$  such that  $Z_{C_s L_l}$  is maximised
- Step 5: Apply  $(C_s L_l)^{-1}$

# Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$l = \arg \max_l Z_{EL_l}.$$

Approximating  $Z_{EL_l}$  therefore allows us to approximate the ML decoder.

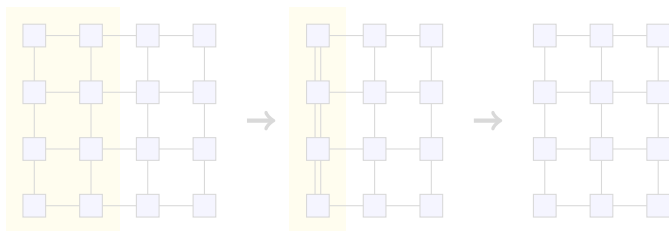
- Step 1: Measure the syndrome  $s$
- Step 2: Construct an arbitrary error  $C_s$  which has syndrome  $s$
- Step 3: Approximate  $Z_{C_s L_l} = \Pr(\overline{C_s L_l})$  for each logical  $l$
- Step 4: Find the  $l$  such that  $Z_{C_s L_l}$  is maximised
- Step 5: Apply  $(C_s L_l)^{-1}$



# Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks<sup>2,3</sup>, allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo<sup>4</sup>:



<sup>4</sup>Verstraete et. al., PRL 2006, doi:10/dfgcz8, arXiv:quant-ph/0601075

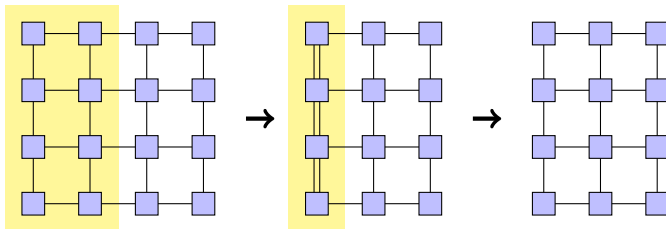
<sup>3</sup>Bridgeman and Chubb, JPA 2017, doi:10/cv7m, arXiv:1603.03039

<sup>4</sup>Bravyi, Suchara, Vargo, PRA 2014, doi:10/cv7n, arXiv:1405.4883

# Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks<sup>2,3</sup>, allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo<sup>4</sup>:



<sup>4</sup>Verstraete et. al., PRL 2006, doi:10/dfgcz8, arXiv:quant-ph/0601075

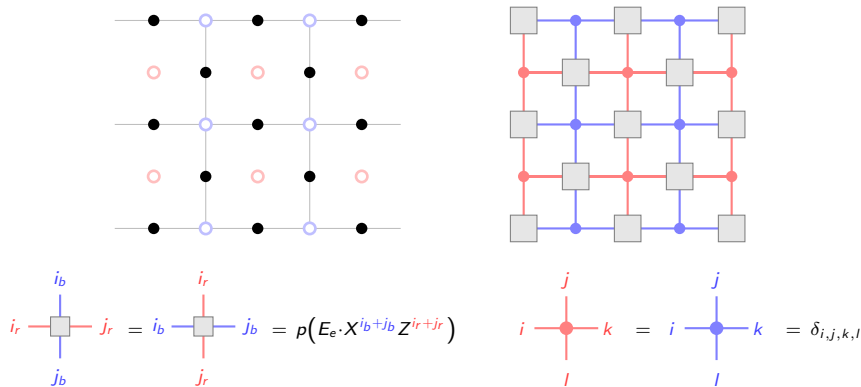
<sup>3</sup>Bridgeman and Chubb, JPA 2017, doi:10/cv7m, arXiv:1603.03039

<sup>4</sup>Bravyi, Scuhara, Vargo, PRA 2014, doi:10/cv7n, arXiv:1405.4883

# Decoding from (approximate) tensor network contraction

This gives an algorithm for (approximate) maximum likelihood decoding for any 2D code, subject to any locally correlated noise, generalising BSV.

Indeed, applying this to iid noise in the surface code reproduces BSV:



# Conclusions and further work

- Extended the stat. mech. mapping to correlated models
  - Can apply stat. mech. mapping to circuit noise via the history code
  - Numerically evaluated the threshold of correlated bit-flips in the toric code
  - Stat. mech. mapping gives tensor network maximum likelihood decoders
- 
- Can we apply this to experimentally relevant correlated models?
  - Construction extends to all Abelian quantum doubles. Can we extend non-Abelian models?
  - Non-stochastic noise? Non-Pauli noise? Coherent noise?
  - Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work with David Tuckett and Benjamin Brown).

**Thank you!**

**ArXiv:1809.10704**

(me@)christopherchubb.com