Moderate deviation analysis for classical communications over quantum channels

Joint work with Vincent Y.F. Tan (NUS)
and Marco Tomamichel (USyd/UTS)
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Classical communication over a quantum channel

We are going to consider transmitting classical information over a quantum channel.

For channel $\mathcal{W}$, a (n, R, ϵ)-code is an encoder E and decoding POVM $\{D_i\}$ such that

$$\frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \mathrm{Tr}\left[\mathcal{W}^{\otimes n}\left(\otimes_{i=1}^{n} E_i(m)\right) D_m\right] \geq 1 - \epsilon$$

We will be concerned with the trade-off between the block-length n, the rate R, and the error probability ϵ. We define the optimal rate/error probability as

$$R^*(\mathcal{W}; n, \epsilon) := \max\left\{R \mid \exists (n, R, \epsilon)\text{-code}\right\},$$
$$\epsilon^*(\mathcal{W}; n, \epsilon) := \min\left\{\epsilon \mid \exists (n, R, \epsilon)\text{-code}\right\}.$$

Asymptotics

For a constant error probability ϵ, the Strong Converse Theorem[1] tells us the rate approaches a constant known as the capacity

$$\lim_{n\to\infty} R^*(\mathcal{W}; n, \epsilon) = C(\mathcal{W}).$$

Equivalently this means that the error probability must to go 0 to 1 either side of the capacity

$$\lim_{n\to\infty} \epsilon^*(\mathcal{W}; n, R) = \begin{cases} 0 & : R < C \\ 1 & : R > C \end{cases}$$

This tells us we can have either $R \to C$ OR $\epsilon \to 0$.

How fast are these convergences? Can we do both?

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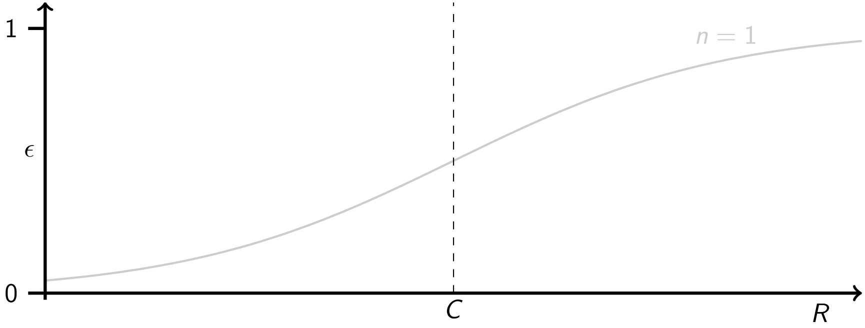
Small and large deviations

How fast are the convergences $R \to C$ or $\epsilon \to 0$ as $n \to \infty$?



Small deviation (Tomamichel & Tan 2015)

$$R^*(n,\epsilon) = C + \sqrt{\frac{V}{n}}\Phi^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right) \quad \epsilon \in (0, \tfrac{1}{2})$$

Large deviation (Partial progress)

$$\ln \epsilon^*(n,R) = -n \cdot E(R) + o(1) \quad R < C$$

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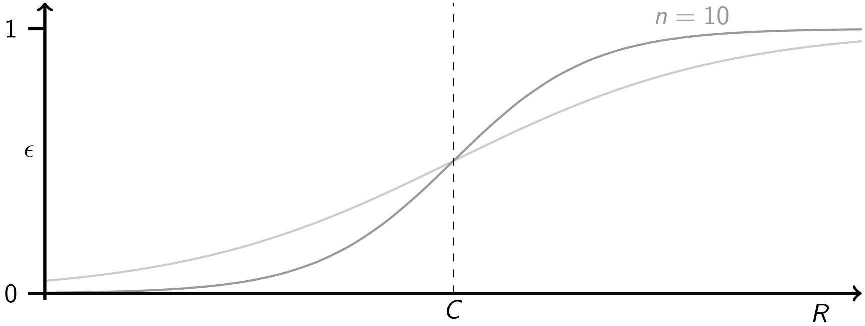

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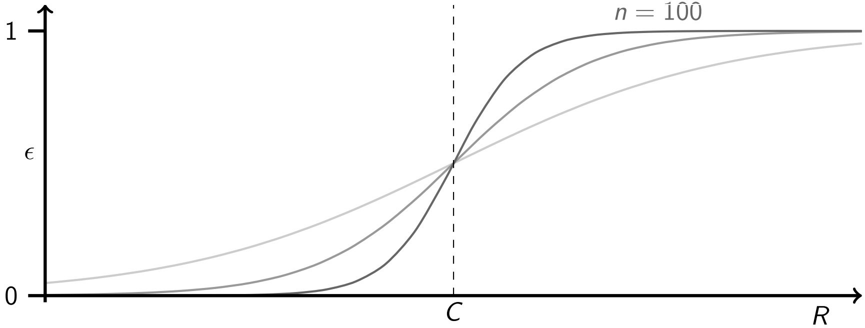

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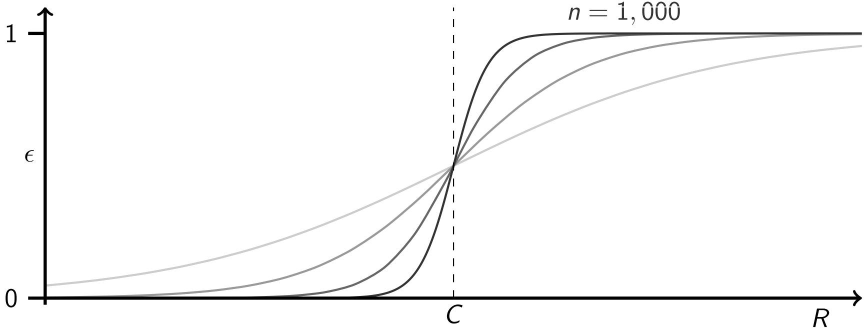

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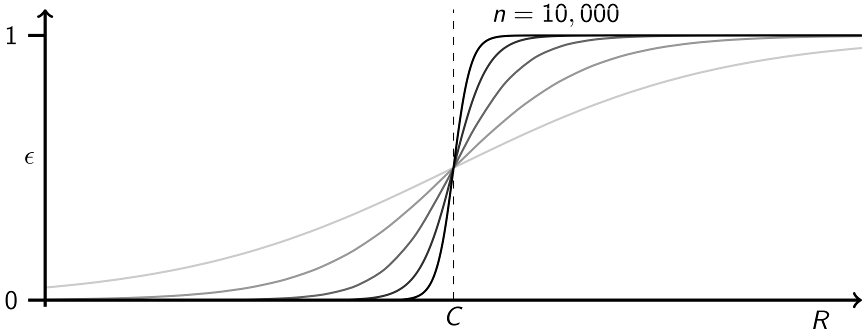

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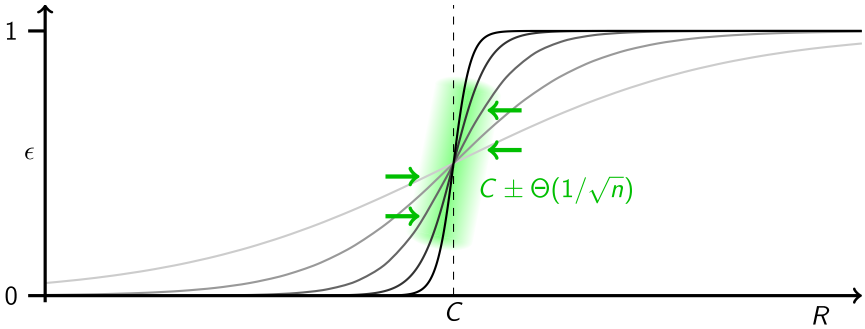

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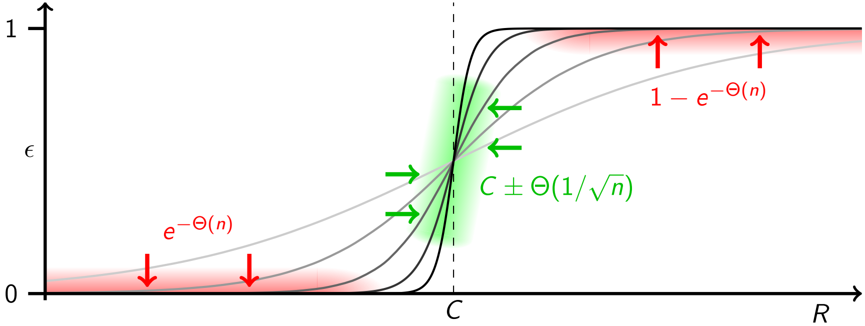

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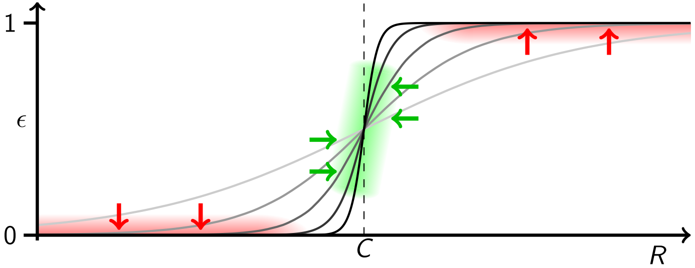

Moderate deviation (This work, Cheng & Hsieh 2017)

For any $\{a_n\}$ such that $a_n \to 0$ and $\sqrt{n}a_n \to \infty$ we have

$$R^*(n, \epsilon_n) = C - \sqrt{2V}a_n + o(a_n) \quad \text{for} \quad \epsilon_n = e^{-na_n^2},$$

or equivalently

$$\ln \epsilon^*(n, R_n) = -\frac{na_n^2}{2V} + o(na_n^2) \quad \text{for} \quad R_n = C - a_n.$$

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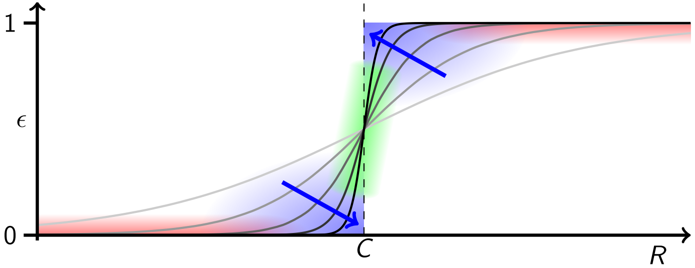

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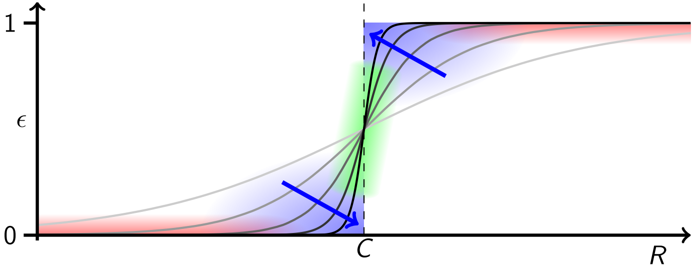

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Concentration inequalities

Take $\{X_i\}$ iid with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}[X_i] =: V$, and $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$.

Asymptotic (Law of large numbers)

$$\lim_{n\to\infty} \Pr\left[\bar{X}_n \geq t\right] = \begin{cases} 1 & t < 0, \\ 0 & t > 0. \end{cases}$$

Small deviation (Berry-Esseen)

$$\Pr\left[\bar{X}_n \geq \frac{\epsilon}{\sqrt{n}}\right] = Q\left(\frac{\epsilon}{\sqrt{V}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad \epsilon \in (0,1)$$

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Hypothesis testing

We want to test between two hypotheses, ρ and σ. For a binary POVM $\{Q, I - Q\}$, we define the type-I and type-II errors as

$$\alpha(Q;\rho,\sigma) := \mathrm{Tr}(I-Q)\rho, \qquad \beta(Q;\rho,\sigma) := \mathrm{Tr}\, Q\sigma,$$

and the ϵ-hypothesis-testing divergence

$$D_h^\epsilon(\rho\|\sigma) := -\log\min\left\{\beta(Q;\rho,\sigma)\,|\,\alpha(Q;\rho,\sigma)\le\epsilon\right\}.$$

If we now consider testing between $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, then the asymptotic behaviour is given by Quantum Stein's Lemma.

Asymptotics (Hiai & Petz 1991 / Ogawa & Nagaoka 1999)

$$\lim_{n\to\infty}\frac{1}{n}D_h^\epsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma) \quad \text{where} \quad \epsilon\in(0,1),$$

or equivalently

$$\frac{1}{n}D_h^{\epsilon_n}(\rho^{\otimes n}\|\sigma^{\otimes n}) = R \quad\Longrightarrow\quad \lim_{n\to\infty}\epsilon_n = \begin{cases}0 & : R < D(\rho\|\sigma),\\ 1 & : R > D(\rho\|\sigma).\end{cases}$$

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Deviation results for hypothesis testing

Small deviation (Tomamichel & Hayashi 2013, Li 2014)

$$\frac{1}{n} D_h^{\epsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) = D(\rho\|\sigma) + \sqrt{\frac{V(\rho\|\sigma)}{n}} \Phi^{-1}(\epsilon) + \mathcal{O}\left(\frac{\log n}{n}\right) \quad \text{for} \quad \epsilon \in (0,1).$$

Large deviation (Hayashi 2006, Nagaoka 2006)

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Reducing hyp. testing to concentration inequalities

To give a moderate deviation analysis of the HTD, we will use concentration bounds. First we see it is related to tail bounds of the Nussbaum-Szkoła distributions[2]

$$P^{\rho,\sigma}(a,b) := r_a |\langle \phi_a | \psi_b \rangle|^2 \quad \text{and} \quad Q^{\rho,\sigma}(a,b) := s_b |\langle \phi_a | \psi_b \rangle|^2,$$

where we have eigendecomposed our states $\rho := \sum_a r_a |\phi_a\rangle\langle\phi_a|$ and $\sigma := \sum_b s_b |\psi_b\rangle\langle\psi_b|$. These reproduce the first two moments of our states

$$D\left(P^{\rho,\sigma} \| Q^{\rho,\sigma}\right) = D(\rho\|\sigma) \qquad \text{and} \qquad V\left(P^{\rho,\sigma} \| Q^{\rho,\sigma}\right) = V(\rho\|\sigma).$$

Specifically for iid $Z_i = \log P^{\rho,\sigma}/Q^{\rho,\sigma}$ and $(a,b) \sim P$, then[3]

$$\frac{1}{n} D_h^{\epsilon_n}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) \geq \sup\left\{ R \,\middle|\, \Pr\left[\sum_{i=1}^n Z_i\right] \leq \epsilon_n/2 \right\} - \mathcal{O}(\log 1/\epsilon_n),$$

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Bounding the rate

For this we can use the one shot bounds

$$R^*(1,\epsilon) \geq \sup_{P_X} D_h^{\epsilon/2}(\pi_{XY}\|\pi_X\otimes\pi_Y) - \mathcal{O}(1), \qquad \text{(Wang \& Renner 2012)}$$

$$R^*(1,\epsilon) \leq \inf_{\sigma} \sup_{\rho\in\mathrm{Im}(\mathcal{W})} D_h^{2\epsilon}(\rho\|\sigma) + \mathcal{O}(1), \qquad \text{(Tomamichel \& Tan 2015)}$$

where $\pi_{XY} = \sum_x P_X(x)\,|x\rangle\langle x|_X \otimes \rho_Y^{(x)}$.

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We show that a moderate deviation analysis of the rate follows from that of the hypothesis testing divergence.

Conclusion and further work

- We have give a moderate deviation analysis of the capacity of c-q channels, and asymmetric binary hypothesis testing.
- Our proof covers the $V = 0$ case which had not been considered in the classical literature.
- This proof nicely extends to image-additive channels (separable encodings) and infinite input alphabets.

- Can we improve the $o(a_n)$ error terms? It seems they might actually be $\mathcal{O}(a_n^2 + \log n)$.
- Can we extend this to a moderate deviation analysis of other capacities such as the quantum or entanglement-assisted?
- What other channels such as entanglement-breaking?

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