From Statistical Mechanical Models to Tensor Network Decoding

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Stat Mech Mapping: arXiv:1809.10704, to appear in AIHPD
with Steve Flammia

Tensor Network decoding: To appear arXiv:2009:?????

Decoders

Passive error correction: physics alone suppress errors

Active error correction: decoder needed to remove error

Two classes of decoder:

- Practical decoders: Speed over accuracy
- Analytic decoders: Accuracy over speed

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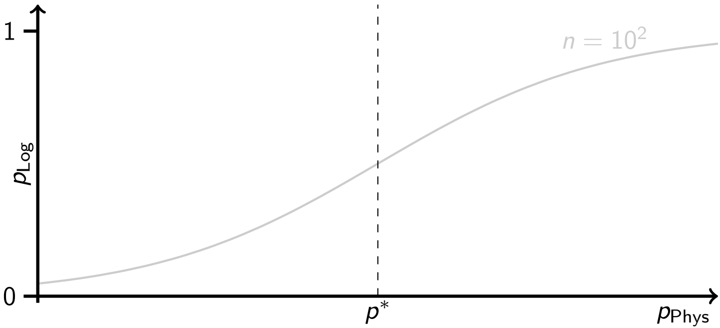
For large system sizes, performance is largely described by the threshold.



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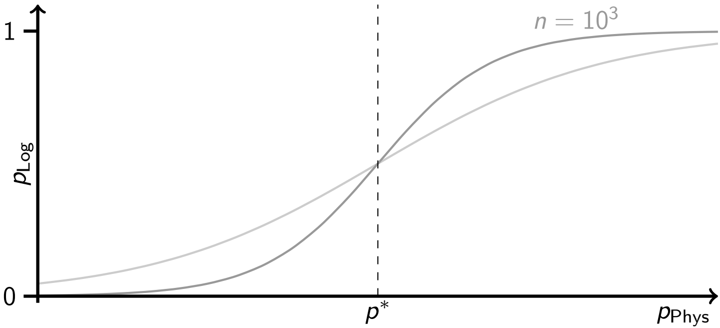

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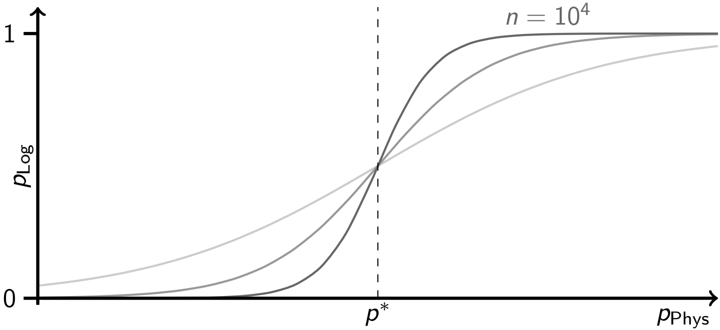

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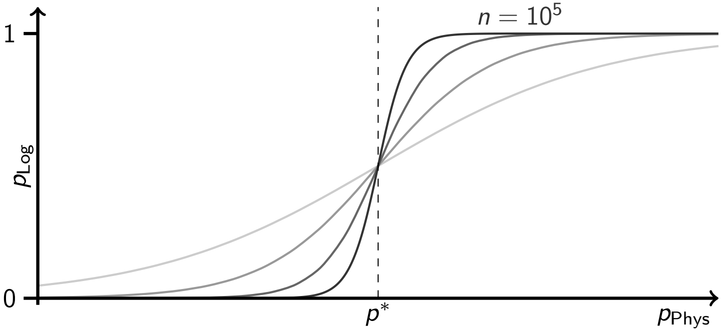

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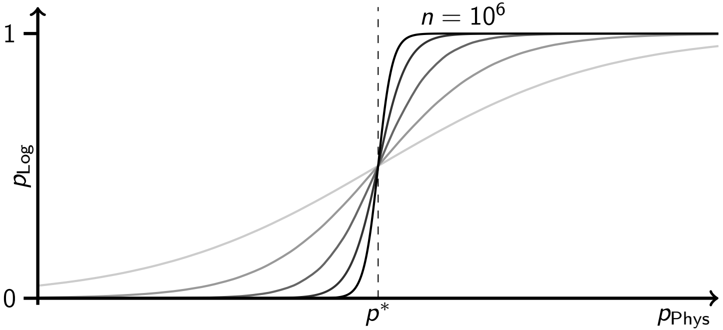

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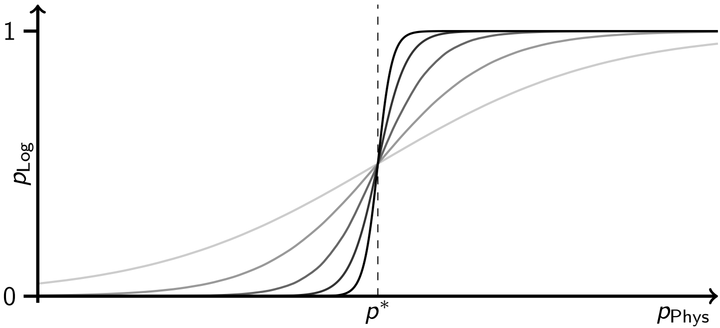

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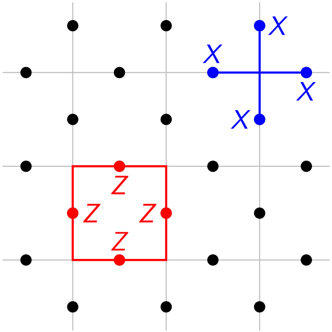

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- Stabilisers form closed loops
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- Errors correspond to open paths
- Syndrome bits corresponds to the ends of paths

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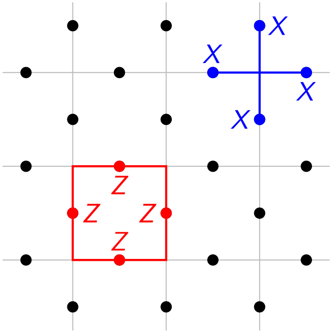

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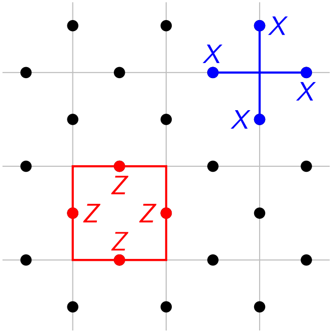

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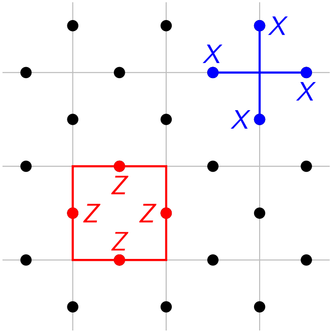

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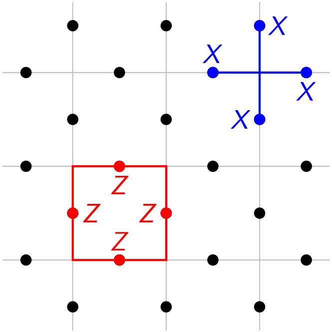

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Pair matching decoder

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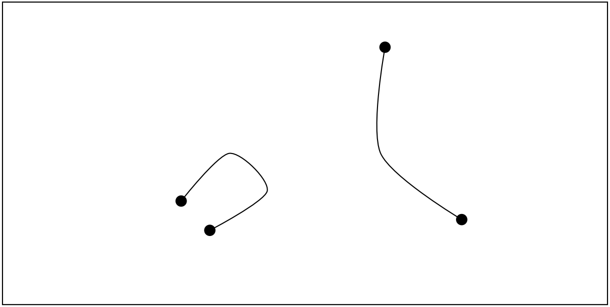

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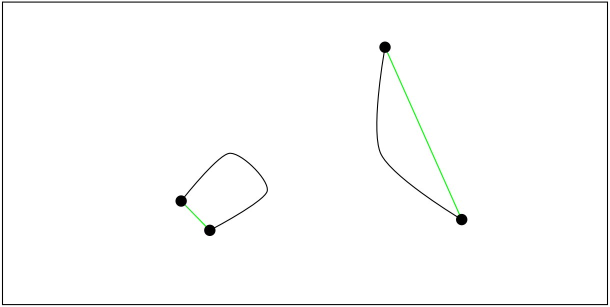

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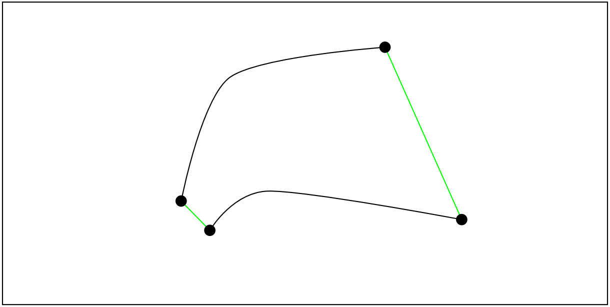

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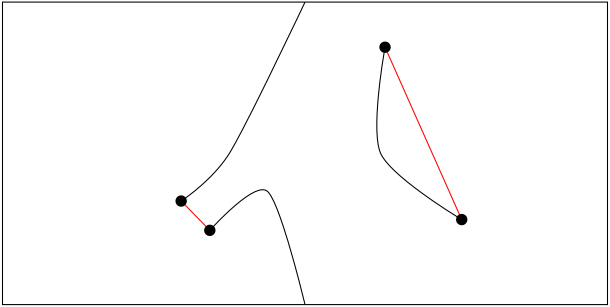

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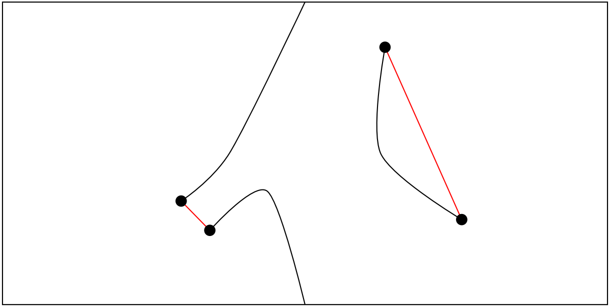

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Most likely error versus maximum likelihood

Why is pair matching suboptimal? In short, degeneracy.

For quantum codes multiple errors can have the same syndrome. A decoder needs to identify the most likely error <u>class</u>, not the single most likely error.

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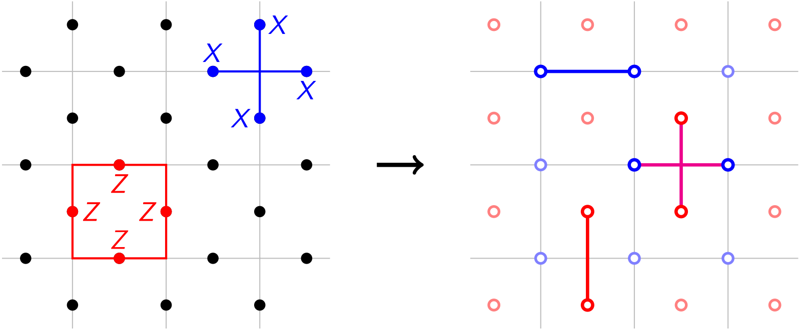
Statistical mechanical mapping

The idea here it to construct a family of statistical mechanical models, whose thermodynamic properties reflect the error correction properties of the code.



Toric code → Eight-vertex model

This will allow us to use the analytic and numerical tools developed to study stat mech systems to study quantum codes.

Statistical mechanical mapping

Stabiliser code & Pauli noise	$\longrightarrow$	Disordered statistical mechanical model
Threshold	$\longleftrightarrow$	Phase transition
Decoding	$\longleftrightarrow$	Calculating partition functions

Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes, e.g.

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Stabiliser codes and Pauli noise

For qubits, the Paulis $\mathcal{P} := \{I, X, Y, Z\}$ are defined

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will be considering stabiliser codes, which are specified by an Abelian subgroup of the Paulis $\mathcal{S}$, and whose code space $\mathcal{C}$ is the joint $+1$ eigenspace,

$$\mathcal{C} = \left\{ |\psi\rangle \,\middle|\, S\,|\psi\rangle = |\psi\rangle\,, \forall S \in \mathcal{S} \right\}.$$

Any two errors which differ by a stabiliser are logically equivalent, so the logical classes of errors are

$$\overline{E} := \{ES | S \in \mathcal{S}\}$$

Independent case: Hamiltonian

Let $[\![A, B]\!]$ be the scalar commutator of two Paulis, such that $AB =: [\![A, B]\!]\, BA$.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E, the (disordered) Hamiltonian H_E is defined

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling}} \overbrace{[\![\sigma, E]\!]}^{\text{Disorder}} \overbrace{\prod_{k : [\![\sigma, S_k]\!] = -1} s_k}^{\text{DoF}}$$

for $s_k = \pm 1$, and coupling strengths $J_i(\sigma) \in \mathbb{R}$.

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis σ
- Disorder E flips some interactions (Ferro $\leftrightarrow$ Anti-ferro)
- Local code $\implies$ local stat. mech. model

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Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

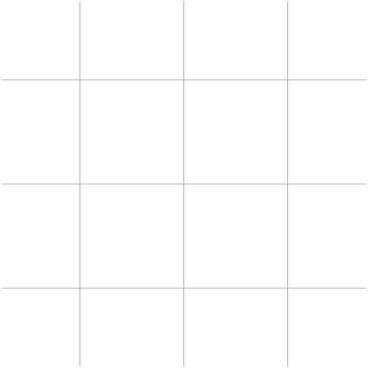
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

where $e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$



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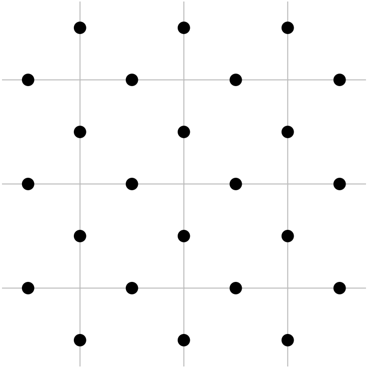

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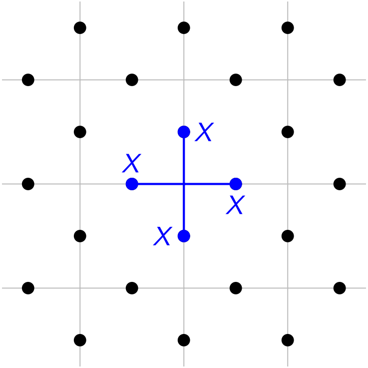

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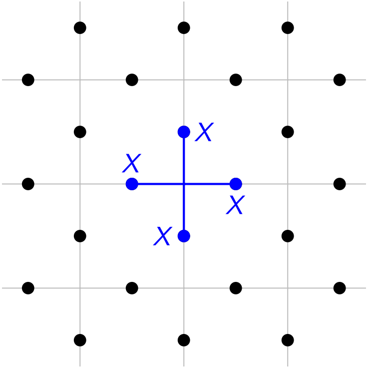

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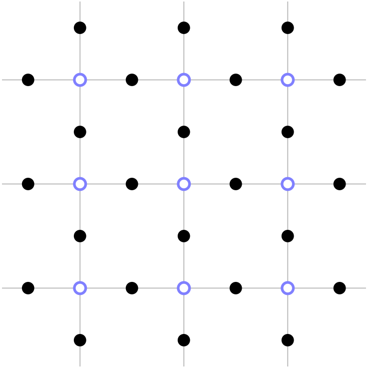

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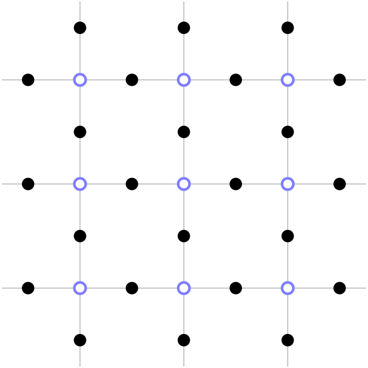

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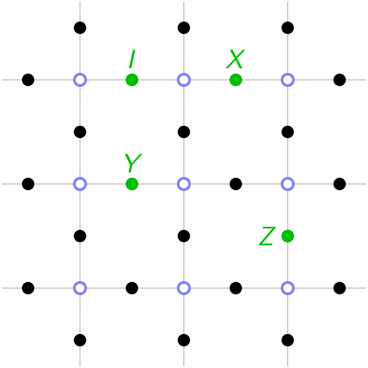

$$[\![\sigma_e, S_v]\!] = \begin{cases} -1 & \text{if } \sigma = Y, Z,\ v \in e, \\ +1 & \text{else.} \end{cases}$$

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Toric code with iid bit-flips

$$\Pr(X_e) = p, \qquad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

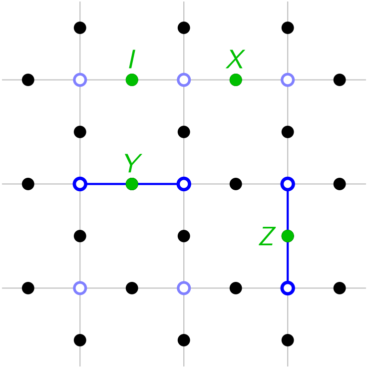
Step 2: Interactions

$$H_I = -\sum_{v \sim v'} J\, s_v s_{v'}$$

Step 3: Disorder

$$H_E = -\sum_{v \sim v'} J e_{vv'}\, s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



$$[\![\sigma_e, S_v]\!] = \begin{cases} -1 & \text{if } \sigma = Y, Z,\ v \in e, \\ +1 & \text{else.} \end{cases}$$

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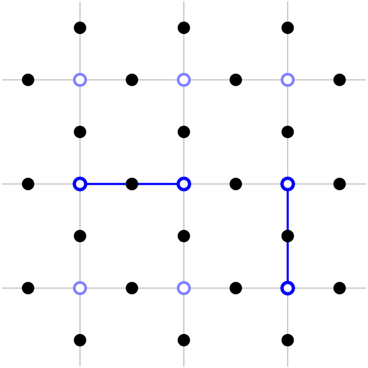

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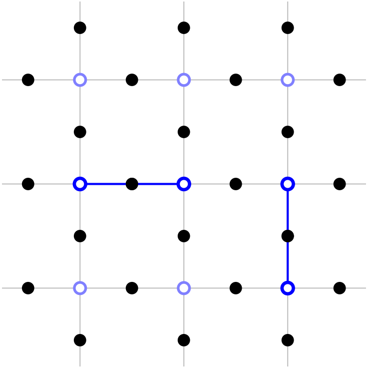

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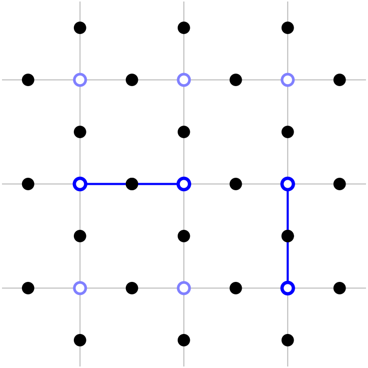

$[\![\sigma, S_k]\!] \;\rightarrow\; [\![\sigma, ES_k]\!]$

Toric code and the random-bond Ising model

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Toric code with iid bit-flips

$\Pr(+J) = p, \quad \Pr(-J) = 1 - p.$

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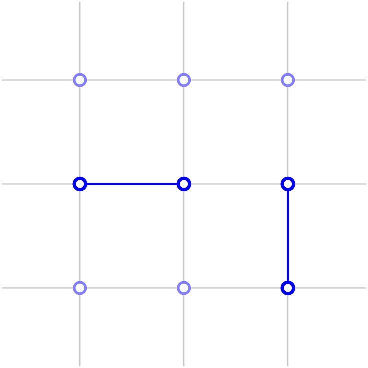
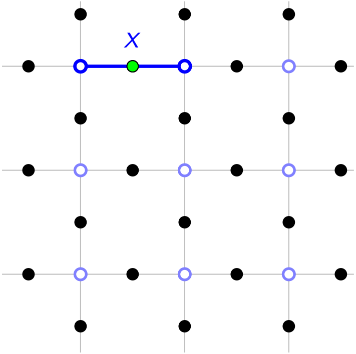


$\pm J$ Random-bond Ising Model

Other independent examples

Toric code

Bit-flip → Random-bond Ising[1]
Indep. $X\&Z$ → 2×Random-bond Ising
Depolarising → Random 8-vertex model[2]



Colour code

Bit-flip → Random 3-spin Ising
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[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
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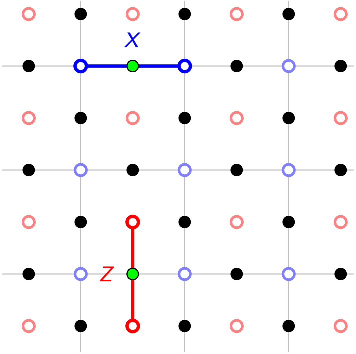

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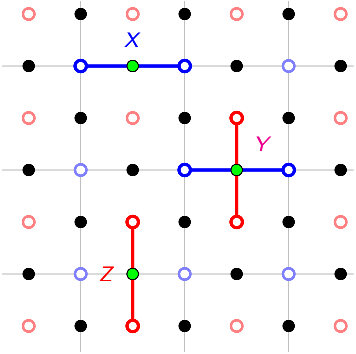

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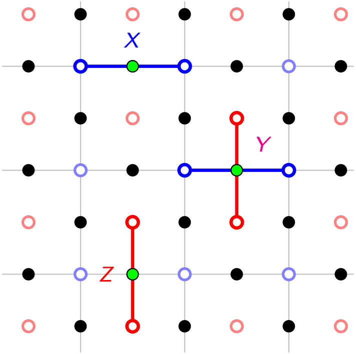

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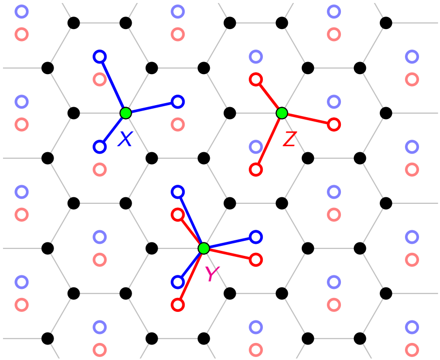

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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error L,

$$\Delta_E(L) = -\frac{1}{\beta}\log Z_{EL} + \frac{1}{\beta}\log Z_E.$$

Along the Nishimori line

$$\Delta_E(L) = \frac{1}{\beta}\log\frac{\Pr(\overline{E})}{\Pr(\overline{EL})},$$

which implies

$$\begin{aligned} \text{Below threshold :} \quad & \Delta_E(L) \to \infty \ \text{(in mean)} \\ \text{Above threshold :} \quad & \Delta_E(L) \to 0 \quad \text{(in prob.)} \end{aligned}$$

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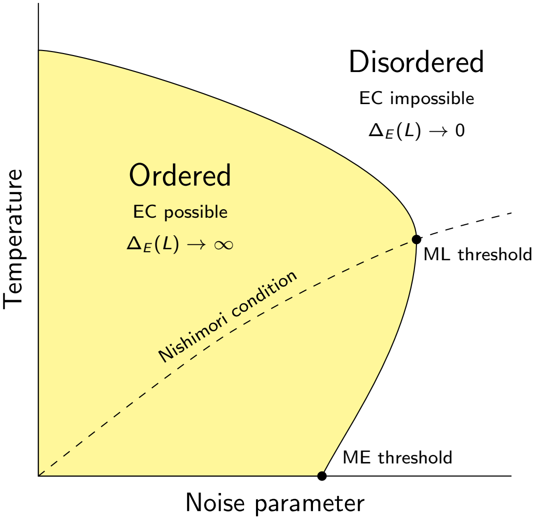
Phase diagram sketch



Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

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This model includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.

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By construction, we can extend to the correlated case by changing $\sigma \in \mathcal{P}_i$ to $\sigma \in \mathcal{P}_{R_j}$:

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Nishimori condition: $$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket ,$$

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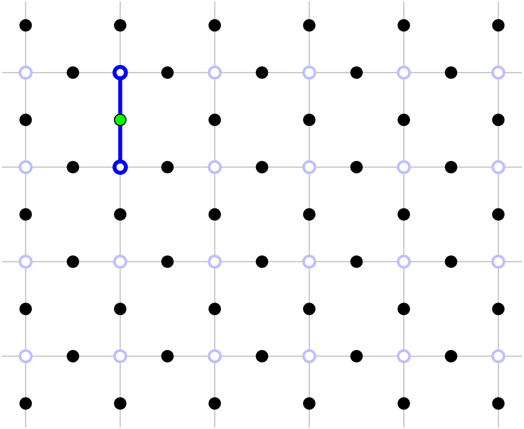
Correlated example

Toric code with correlated bit-flips

Correlations induce longer-range interactions



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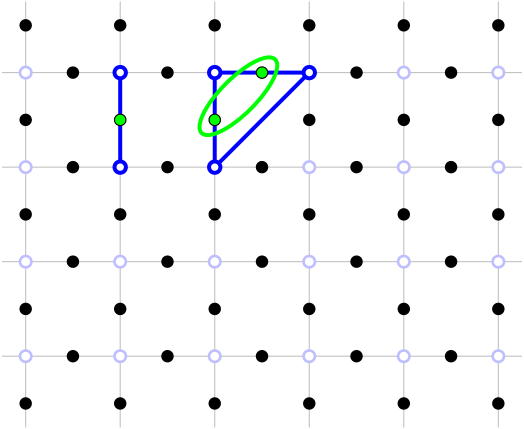

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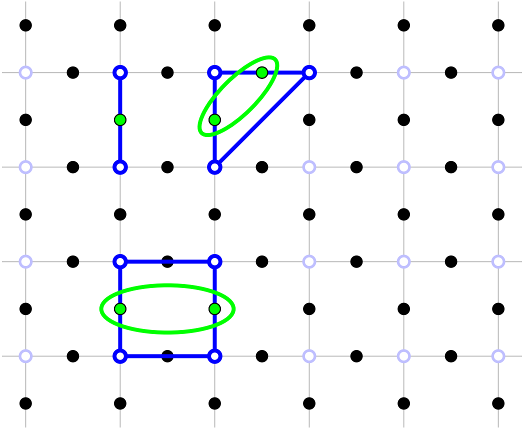

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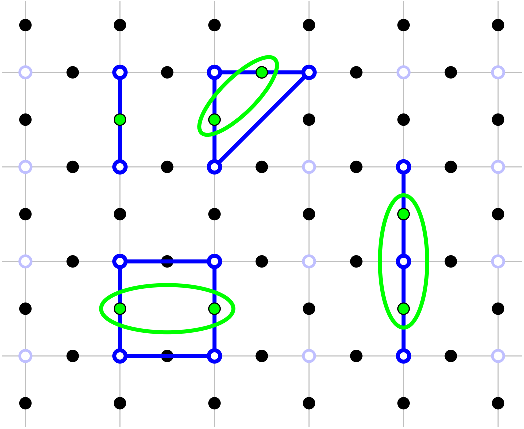
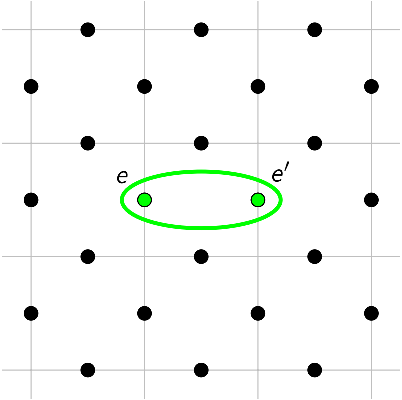


'Across plaquette' correlated bit-flips



This error model is entirely specified by the conditional error probabilities

$$
\begin{array}{ll}
\Pr(I_e|I_{e'}) & \Pr(I_e|X_{e'}) \\
\Pr(X_e|I_{e'}) & \Pr(X_e|X_{e'})
\end{array}
$$

for all neighbouring edges e and e'.

For our purposes, it will convenient to parameterise things by

$$
p := \Pr(X_e), \qquad \eta := \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}.
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Here p is the marginal error rate, and η is a measure of the correlations.

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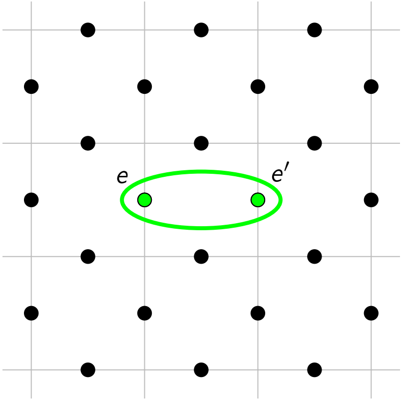

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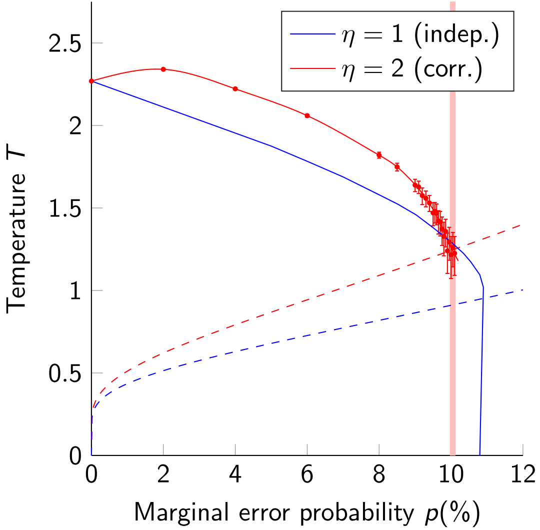
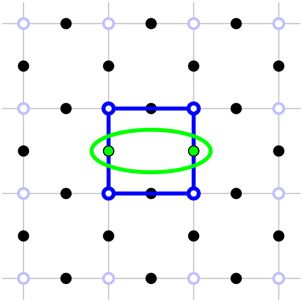
Monte Carlo simulations





Thresholds

Indep.: $p_{\mathrm{th}} = 10.917(3)\%$[1,2]
Corr.: $p_{\mathrm{th}} = 10.04(6)\%$

---

[1] Dennis et.al., JMP 2002, `doi:10/cs2mtf`, `arXiv:quant-ph/0110143`
[2] Toldin et.al., JSP 2009, `doi:10/c3r2kc`, `arXiv:0811.2101`

Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$\ell = \arg\max_{\ell} Z_{EL_\ell}.$$

Approximating Z_{EL_l} therefore allows us to approximate the ML decoder.

- Step 1: Measure the syndrome s
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Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks[1,2], allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo[3]:

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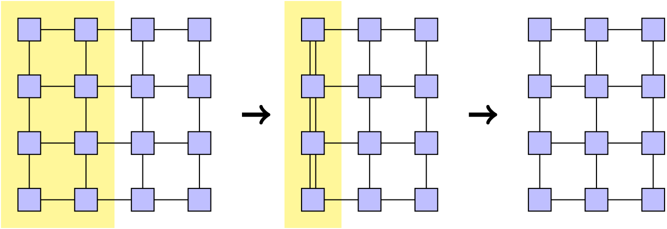

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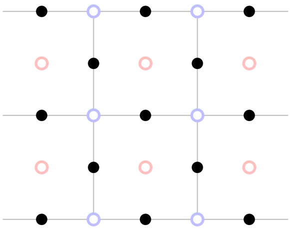
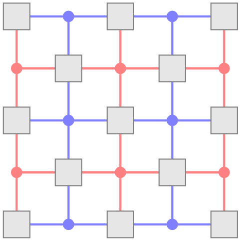
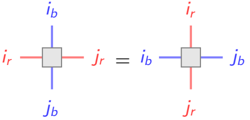
Decoding from (approximate) tensor network contraction

This gives an algorithm for (approximate) maximum likelihood decoding for any 2D code, subject to any locally correlated noise, generalising BSV.

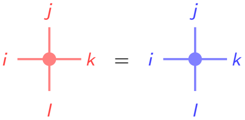
Indeed, applying this to iid noise in the surface code reproduces BSV:







$= p\left(E_e \cdot X^{i_b+j_b} Z^{i_r+j_r}\right)$



$= \delta_{i,j,k,l}$

Ongoing work: Surface codes on different graphs

The TN decoder lets us efficient probe the threshold of 2D topological codes.
What happens if we change the underlying graph?

Raise or lower the connectivity

Irregular graphs

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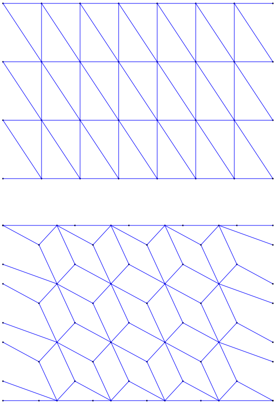

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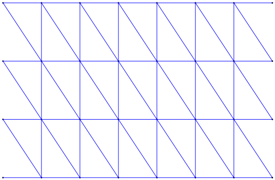

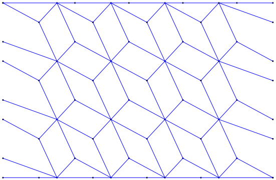

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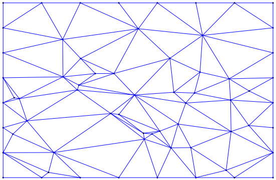
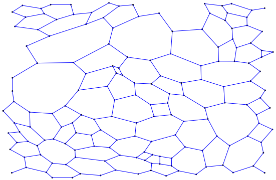




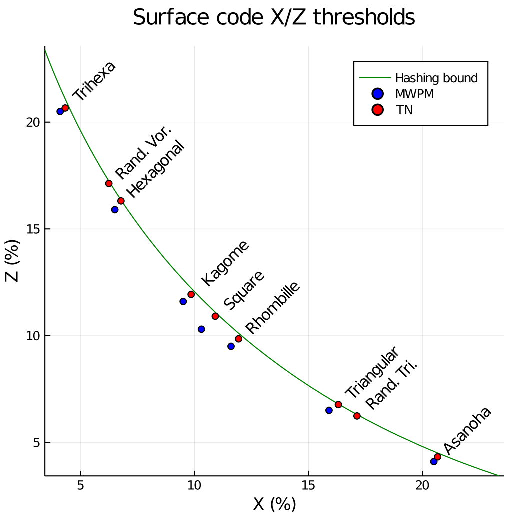
Ongoing work: Surface codes on different graphs

We find a trade-off between the X and Z thresholds.

Hashing bound: $h(p_x) + h(p_z) < 1$.

Pair matching studied earlier by Fujii et.al.[4]

We are currently running similar numerics for depolarising noise, and the colour code.



[4] Fujii et.al., `doi.org/d5sb`, `arXiv:1202.2743`

Further work

- TN decoding of LDPCs (ongoing work with Stefanos Kourtis)
- Use TN decoder to design codes for correlated noise

Thank you!

Stat Mech Mapping: arXiv:1809.10704, to appear in AIHPD
Tensor Network decoding: To appear arXiv:2009:?????

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`@QuantumChubb`